

SAMPLE MATH 55 MIDTERM 1, SPRING 2018

- (1) Mark each of the following questions true (T) or false (F). Provide a sentence or two justifying each answer.

(a) If  $x \equiv y \pmod{m}$  then  $ax \equiv ay \pmod{m}$ . T

$$x \equiv y \pmod{m} \Rightarrow m \mid (x-y) \Rightarrow m \mid a(x-y) \Rightarrow m \mid ax - ay \Rightarrow ax \equiv ay \pmod{m}$$

(b) If  $ax \equiv ay \pmod{m}$  then  $x \equiv y \pmod{m}$ . F

$$3 \cdot 2 \equiv 3 \cdot 4 \pmod{6} \text{ but } 2 \not\equiv 4 \pmod{6}$$

$$a \cdot x \quad a \cdot y \qquad \qquad x \neq y$$

(c) The function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  defined by  $f(x) = \lfloor \frac{x}{2} \rfloor$  is surjective. T

For each integer  $z$  in the codomain,  
 $f(2z) = z$ .

(d) The set of integers and the set of even integers have the same cardinality. T

bijection  $f : \mathbb{Z} \rightarrow \text{Even integers}$  is  
 $f(x) = 2x$

(e) The positive real numbers are countable. F

We showed that the interval  $[0, 1]$  is uncountable.

(f) Let  $\mathbb{R}$  be the domain, and let  $P(x, y)$  be the statement  $y^2 = x$ . Determine the truth value of the following statement:

$$\forall x \exists y P(x, y).$$

Choose  $x = -1$ . Then  $\nexists y \in \mathbb{R}$  s.t.  $y^2 = -1$ .

- (2) Prove that if  $m$  is a positive integer of the form  $4k+3$  for some non-negative integer  $k$ , then  $m$  is not the sum of the squares of two integers.

(We did this in Lecture 6.)

Lem: Show that if  $n$  is an integer, then  $\frac{n^2}{n^2} \equiv 0$  or  $1 \pmod{4}$ .

Pf:  $n$  is either odd or even.

If odd,  $n = 2k+1$  for  $k \in \mathbb{Z}$ .

So  $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 \equiv 1 \pmod{4}$ .

If even,  $n = 2k$  for  $k \in \mathbb{Z}$ .

So  $n^2 = (2k)^2 = 4k^2 \equiv 0 \pmod{4}$ . □

Thm: If  $m$  is pos. integer of the form  $4k+3$  for some non-neg integer  $k$ , then  $m$  is not the sum of the squares of two integers.

Pf: Assume, for sake of  $\Rightarrow \Leftarrow$ , that  $m = a^2 + b^2$  where  $a, b \in \mathbb{Z}$ .

Then by previous exercise,  $a^2, b^2 \equiv 0$  or  $1 \pmod{4}$ .

So  $m = a^2 + b^2 \equiv 0$  or  $1$  or  $2 \pmod{4}$ .

But  $4k+3 \equiv 3 \pmod{4}$ .

So this was a contradiction  $\Rightarrow \Leftarrow$  □

(3) Prove that  $\overline{A \cup B} = \overline{A} \cap \overline{B}$ .

Need to show  $\textcircled{1} \quad \overline{A \cup B} \subset \overline{\overline{A} \cap \overline{B}}$  and  
 $\textcircled{2} \quad \overline{\overline{A} \cap \overline{B}} \subset \overline{A \cup B}$

$\textcircled{1}$  Suppose  $x \in \overline{A \cup B}$ .

Then  $x \notin A \cup B$ .

So  $x \notin A$  and  $x \notin B$ .

So  $x \in \overline{A}$  and  $x \in \overline{B}$

$\therefore x \in \overline{A} \cap \overline{B}$ .

$\therefore \overline{A \cup B} \subset \overline{\overline{A} \cap \overline{B}}$ .

$\textcircled{2}$  Suppose  $x \in \overline{\overline{A} \cap \overline{B}}$ .

Then  $x \in \overline{A}$  and  $x \in \overline{B}$ .

So  $x \notin A$  and  $x \notin B$ .

$\therefore x \notin A \cup B$ .

So  $x \in \overline{A \cup B}$ .

$\therefore \overline{\overline{A} \cap \overline{B}} \subset \overline{A \cup B}$ .

(4) Computation.

- Write the number 466 in base 9.

$$466 = 9 \cdot 51 + 7$$

$$51 = 9 \cdot 5 + 6$$

$$5 = 9 \cdot 0 + 5$$

$$\therefore 466 = (567)_9.$$

$$\text{Check: } 5 \cdot 9^2 + 6 \cdot 9 + 7 = 466 \quad \checkmark$$

- What is the sum of the first  $n$  entries of the sequence 1, 3, 9, 27, 81, ...?

$$a + a \cdot r + ar^2 + ar^3 + \dots + ar^l = a \cdot \left( \frac{r^{l+1} - 1}{r - 1} \right)$$

$$\text{Here } a = 1, r = 3.$$

$$\text{So } 1 + 3 + 3^2 + \dots + 3^l = \frac{3^{l+1} - 1}{2}.$$

$$\text{So sum of first } n \text{ entries is } \frac{3^n - 1}{2}.$$

- Calculate  $(47^{100} + 25^4) \bmod 23$ .

$$47 \bmod 23 = 1.$$

$$25 \bmod 23 = 2.$$

$$\text{So get } 1^{100} + 2^4 \bmod 23 =$$

$$1 + 16 \bmod 23 = \underline{\underline{17 \bmod 23}}.$$

- What is the contrapositive of the statement "If my cell phone rings, I disturb the lecture."

If I don't disturb the lecture, my cell phone doesn't ring.

- (5) Prove that if  $p$  is prime, the only solutions of  $x^2 \equiv 1 \pmod{p}$  are integers  $x$  such that  $x \equiv 1 \pmod{p}$  or  $x \equiv -1 \pmod{p}$ .

We know from Lecture 7 that

Lem: If  $p$  prime, and  $p \mid a_1 a_2 \dots a_n$   
then  $p \mid a_i$  for some  $i$ .

(came from statement that if  $a \mid bc$  and  $\gcd(a,b)=1$  then  $a \mid c$ )  
which we proved using Bezout

$$x^2 \equiv 1 \pmod{p} \Rightarrow$$

$$p \mid (x^2 - 1) \Rightarrow$$

$$p \mid (x-1)(x+1) \Rightarrow$$

$$p \mid (x-1) \text{ or } p \mid (x+1).$$

$$\therefore x \equiv 1 \pmod{p} \text{ or } x \equiv -1 \pmod{p}.$$

- (6) Show that if  $a$  and  $b$  are positive integers, then  $ab = \gcd(a, b) \cdot \text{lcm}(a, b)$ .

Write  $a = p_1^{k_1} \cdots p_m^{k_m}$  the prime factorization.  
 $b = p_1^{l_1} \cdots p_m^{l_m}$  " "

where  $k_1, \dots, k_m, l_1, \dots, l_m \in \mathbb{Z}_{\geq 0}$ .

Then know  $\gcd(a, b) = p_1^{\min(k_1, l_1)} \cdots p_m^{\min(k_m, l_m)}$   
 $\text{lcm}(a, b) = p_1^{\max(k_1, l_1)} \cdots p_m^{\max(k_m, l_m)}$ .

So  $\gcd(a, b) \text{lcm}(a, b) =$

★  $p_1^{\min(k_1, l_1)} p_1^{\max(k_1, l_1)} \cdots p_m^{\min(k_m, l_m)} p_m^{\max(k_m, l_m)}$ .

For any integers  $k, l$  either

$$\min(k, l) = k \text{ and } \max(k, l) = l \quad \text{or}$$

$$\min(k, l) = l \text{ and } \max(k, l) = k.$$

So  $p_1^{\min(k_1, l_1)} p_1^{\max(k_1, l_1)} = p_1^{k_1} p_1^{l_1}$ .

Same for  $p_2, \dots, p_m$ .

$$\begin{aligned} \therefore \star &= p_1^{k_1} p_1^{l_1} \cdots p_m^{k_m} p_m^{l_m} \\ &= a \cdot b. \end{aligned}$$

∴  $\gcd(a, b) \text{lcm}(a, b) = ab$ .