Recall: Let $A, B$ be sets. A binary relation $R$ from $A$ to $B$ is a relation on $A$.

A relation $R$ on $A$ is transitive if

Suppose $R$ is not trans. What do we add to $R$ to make it trans?

Say $R = \{1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 4, 4 \rightarrow 5\}$ What to add?

Def: A path of length $n$ from $a$ to $b$ in the directed graph $G$ is a
Also: there is path from a to b in relation R if

Recall: If R relation on A,
\[ R \circ R = \{(a, c) \} \]
So \[ R \circ R \circ R = \{(a, b) \} \]

Notation: \( R^2 = R \circ R, \quad R^3 = R \circ R \circ R, \ldots \)

Theorem: Let R be relation on A.
There is path of length n,

Def: Let R be relation on A. The connectivity relation \( R^* \) is
\[ R^* = \{(a, b) \} \]
So \( R^* = \)

Ex: Let \( R = \{(a, b) \} \) there is simple cycle blur from place a to b
\[ R^* = \]
Thm: The transitive closure of relation $R$ is
$R^*$.

Proof: What do we need to show?
$R^*$ is the smallest transitive relation that contains $R$

(1)

(2)

(3)

Section 9.5 Equivalence Relations

Definition: A relation $R$ on set $A$ called an equivalence relation if $R$ is reflexive and symmetric.

Recall: $R$ is reflexive if $aRa$ for all $a$ in $A$
$R$ is symmetric if $$(a,b) \in R \Rightarrow (b,a) \in R$$

Example: Choose $n \in \mathbb{Z}^+$ and let $R$ be relation on $\mathbb{Z}$ defined by

$R =$
**Def:** If $(a,b) \in R$ and $R$ an equiv. relation, we say $a$ and $b$ are *alms*.

Sometimes use notation $\sim$ to denote equiv. relation.

**Ex:**

**Ex:** Are these equiv. relations on $\{0,1,2\}$?

- $\{(0,0), (1,1), (0,1), (1,0)\}$
- $\{(0,0), (1,1), (2,2), (0,1), (1,2)\}$
- $\{(0,0), (1,1), (2,2), (0,1), (1,2), (1,0), (2,1)\}$
- $\{(0,0), (1,1), (2,2), (0,1), (0,2), (1,0), (1,2), (2,0), (2,1)\}$
- $\{(0,0), (1,1), (2,2)\}$

**Ex:** Which of these relations on set of all functions $\mathbb{Z} \to \mathbb{Z}$ are equiv. relations?

- $R = \{(f,g) \mid f(1) = g(1)\}$
- $R = \{(f,g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$
Ex: Let $R$ be relation on the set $\mathbb{Z} \times \mathbb{Z} = \{(a, b) \mid a, b \in \mathbb{Z}^+\}$ s.t. 
$((a, b), (c, d)) \in R$ iff $a + d = b + c$. Show that 
$R$ is equiv. relation.

Def: Let $R$ be an equiv. relation on set $A$. 
Choose $a \in A$. Define
Ex: What is equiv. class of \([1,2]\) for congruence mod 5? Let \(R = \{(a,b) \mid a = b \mod 5\}\)

Ex: How many equiv. classes are there mod 5?

SOL:

Note:

Theorem: Let \(R\) be an equiv. relation on set \(A\).

TFAE:
Theorem: Let $R$ be equiv. relation on set $A$. Then: The union of all equiv. classes is $A$. Two equiv. classes are either equal or disjoint (empty intersection).

Def: A partition of a set $S$ is

Ex: