Recall: Let $A,B$ be sets. A binary relation $R$ from $A$ to $B$ is a relation on $A$ is transitive if whenever $a R b$ and $b R c$, then $a R c$.

Suppose $R$ is not trans. What do we add to $R$ to make it trans?

Say $R =$

Attempt 1:

Is it enough to add a new arrow $a \rightarrow c$ whenever $a \rightarrow b$ and $b \rightarrow c$?

Attempt 2:

Def: A path of length $n$ from $a$ to $b$ in the directed graph $G$ is a sequence of edges...
Also: there is path from $a$ to $b$ in relation $R$ if there is sequence of elements

What are paths of length 2

Recall: If $R$ relation on $A$,

$R \circ R = \{(a, c) | \exists b : (a, b) \in R \land (b, c) \in R\}$

So $R \circ R \circ R = \{(a, d) | \exists b, c : (a, b) \in R \land (b, c) \in R \land (c, d) \in R\}$

Notation: $R^2 = \cdots, R^3 = \cdots$

Theorem: Let $R$ be relation on $A$.
There is path of length $n$, where $n \in \mathbb{Z}^+$, from $a$ to $b$, iff

Definition: Let $R$ be relation on $A$, The connectivity relation $R^* = \{(a, b) \mid \exists \text{ path of length } n \}\$?

So $R^* = \{(a, b) \mid \exists \text{ path of length } n \}$

Example: Let $R = \{(a, b) \mid \}\$

$R^* = ??$
Thm: The transitive closure of relation $R$ is $R^*$

Proof: What do we need to show?
"$R^*$ is the smallest transitive relation that contains $R$"

(1)

(2)

(3)

(i) $R^* \supseteq R$ by def.

(ii) To show $R^*$ is transitive: need to show that

(iii)

of 9.5 Equivalence Relations

Def: A relation $R$ on set $A$ called

equivalence relation if

Recall: $R$ is reflexive if $\forall a \in A$

$R$ is symmetric if whenever $(a,b) \in R$, we also have

Ex: Choose $n \in \mathbb{Z}^+$ let $R$ be relation on $\mathbb{Z}$ defined by

$R = \{(a,b) | a - b \equiv 0 \mod n\}$

Then $R$ is

We showed
Def: If \((a, b) \in R\) and \(R\) an equiv. relation, we say \(a\) and \(b\) are

Sometimes use notation

\[ \begin{align*}
\text{Ex: } R = \{(a, b) \mid a \equiv b \mod n\} \text{ is} \\
& \text{we use notation}
\end{align*} \]

\[ \text{Ex: Are these equiv. relations on } \{0, 1, 2\} \ ? \]

\[ \text{Ex: Which of these relations on set of all} \]

\[ \text{functions } \mathbb{Z} \to \mathbb{Z} \text{ are equiv. relations?} \]

* \(R = \{(f, g) \mid \) \]
  Reflex: 
  Symm: 
  Trans: 

* \(R = \{(f, g) \mid \) \]
  Reflex: 
  Symm: 
  Trans: 
Ex: Let \( R \) be a relation on the set \( \mathbb{Z} \times \mathbb{Z} = \{ (a, b) \mid a, b \in \mathbb{Z}^+ \} \) such that \((c, d), (e, f) \in R \) iff 

Def: Let \( R \) be an equivalence relation on set \( A \). Choose \( a \in A \). Define \( \left[ a \right]_R \) called the equivalence class of \( a \). Sometimes denoted \( [a] \). Any element \( b \in \left[ a \right]_R \) called a
Ex: What is equiv. class of 1, 2, for

Ex: How many equiv. classes are there

Sol:

Note:

Theorem: Let $R$ be an equiv. relation on set $A$.

TFAE: (i)  (ii)  (iii)

Pf: Show (i) $\Rightarrow$ (ii).
Show (ii) ⇒ (iii)

Show (iii) ⇒ (i).

**Theorem**: Let R be equiv. relation on set A.
Then: The union of all equiv. classes is
Two equiv. classes are either

**Def**: A partition of a set S is collection of disjoint nonempty subsets of S.

**Ex**: 

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