SAMPLE MATH 55 MIDTERM 2, SPRING 2018

(1) Mark each of the following questions true (T) or false (F). You do not need to justify your answers.

(a) If X and Y are independent random variables, then $E(XY) = E(X) + E(Y)$.

F

(b) In a group of five people, where each two are either friends or enemies, there must be either three mutual friends, or three mutual enemies.

F

(c) If X is a random variable on the sample space S, then $X(s) \geq 0$ for all $s \in S$.

F

(d) If a is an integer and m is a positive integer, then $a^{m-1} \equiv 1 \bmod m$.

F

(e) Let m be a positive integer, and $a_1, a_2, \ldots, a_n$ be integers. If m divides $a_1 a_2 \ldots a_n$, then m divides a_i for some i.

F

(f) If $f : X \to Y$ is a surjective function and $g : Y \to Z$ is a surjective function, then the composition $g \circ f : X \to Z$ is a surjective function.

T

(2) (a) Let X be a random variable, and $a \in \mathbb{R}$. Show that $V(aX) = a^2V(X)$.

Recall: $V(X) = \sum_{s \in S} (X(s) - E(X))^2 \, p(s)$.

$$V(aX) = \sum_{s \in S} (aX(s) - E(aX))^2 p(s)$$
$$= \sum_{s \in S} (aX(s) - aE(X))^2 p(s)$$
$$= \sum_{s \in S} a^2 (X(s) - E(X))^2 p(s).$$
$$= a^2 V(X).$$

(b) If X and Y are two independent random variables on the sample space S, then $V(X+Y) = V(X) + V(Y)$.

Recall: $V(X) = E(X^2) - E(X)^2$.

$$V(X+Y) = E((X+Y)^2) - E(X+Y)^2$$
$$= E(X^2 + 2XY + Y^2) - E(X+Y)^2$$

linearity of expectation

$$= E(X^2) + 2E(XY) + E(Y^2) - (E(X) + E(Y))^2$$

Since X and Y indep.

$$= E(X^2) + 2E(X)E(Y) + E(Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2$$
$$= E(X^2) - E(X)^2 \quad + E(Y^2) - E(Y)^2$$
$$= V(X) \quad + V(Y).$$

$\boxed{///}$

(3) Consider all permutations of the letters ABCDEFG.
   (a) How many of these permutations contains the strings ABC and DE (each as consecutive substrings)?
   (b) In how many permutations does A precede B? (not necessarily immediately)

(a) Need to permute 4 "letters":
ABC, DE, F, G.
$4! = 24$

(b) A precedes B in half of the 7! permutation.
$\therefore \frac{1}{2} \cdot 7! = 2520.$

(4) A store gives out gift certificates in the amounts of \$10 and \$25. What amounts of money can you make using gift certificates from the store? Prove your answer using strong induction.

10, 20, 25, 30, 35, 40, 45, 50, ...

Answer: can make \$n where
$n \in \{10\} \cup \{5m \mid m \geq 4 \text{ and } m \in \mathbb{Z}^+\}$
(\$10 plus any multiple of \$5 starting w/ \$20)

Let $P(m)$ be the statement
"We can make \$5m in gift certificates in amount of \$10 and \$25."

Base Case: $m = 4, 5$.
Clearly we can make \$20 and \$25 in gift certificates. ✓

Inductive Hyp: We can make \$5k for $4 \leq k < m$.
Want to prove $P(m)$, for $m \geq 6$.

Note that (★) $5m = 10 + 5(m-2)$.
Since $4 \leq m-2 < m$, $P(m-2)$ is true.
So we can make \$5(m-2) in gift certificates.
$\therefore$ (★) $\Rightarrow$ can make \$5m in gift certificates by adding an extra \$10 certificate.

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(5) A thumb tack is tossed until it first lands with its point down, at which time no more tosses are made. On each tack toss, the probability of landing point down is 1/3.

(a) Find the probability that exactly five tosses are made.

Exactly 5 tosses are made if the thumb tack lands as follows:
up, up, up, up, down

probability of this sequence is:

$$\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{2^4}{3^5} = \frac{16}{243}$$

(b) What is the expected number of tosses?

tosses = RV w/ geometric dist w/ parameter 1/3.

$\therefore$ expected # tosses is 3.

Alternatively, if you don't remember that stuff, note that the prob of tossing exactly n times is: $\frac{2^{n-1}}{3^n}$.
So then you need to compute

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{n-1}} \cdot n = 3$$

(6) Prove that for every positive integer n, there are n consecutive composite integers. (Hint: consider the n consecutive integers starting with $(n+1)!+2$.)

Consider the n consecutive integers

$$\underbrace{(n+1)!+2}_{\text{divisible by } 2},\ \underbrace{(n+1)!+3}_{\text{divisible by } 3},\ \underbrace{(n+1)!+4}_{\text{divisible by } 4},\ \ldots,\ \underbrace{(n+1)!+(n+1)}_{\text{divisible by } (n+1).}$$

$\therefore$ each of these numbers is composite.

$\blacksquare$