MATH 249 PROBLEM SET 5 (DUE THURSDAY APR. 27)

(1) Prove (using only the definition of representations) that the symmetric group S_n, $n \geq 2$, has exactly two one-dimensional representations: the trivial representation and the sign representation.

(2) Let G be a finite group and let V and W be two representations of G. If ϕ is a G-linear map from V to W, show that $\ker\phi$, $\operatorname{im}\phi$, and $\operatorname{coker}\phi$ are also representations of G.

(3) Let G by the cyclic group of order 3, i.e. $G = \{x \mid x^3 = 1\}$, and let V be the two-dimensional G-module with basis v_1, v_2, where

$$x \cdot v_1 = v_2, \quad x \cdot v_2 = -v_1 - v_2.$$

(a) Show that V is indeed a G-module and write out explicitly the homomorphism $\rho : G \to \mathrm{Gl}_2(\mathbb{C})$.
(b) Express V as a direct sum of irreducible G-modules.

(4) Let G be a finite group and let $\rho : G \to \mathrm{Gl}_2(\mathbb{C})$ be a representation of G. Suppose that there are elements g, h in G such that the matrices $\rho(g)$ and $\rho(h)$ do not commute. Prove that ρ is irreducible.

(5) Let λ be the partition $(n-1, 1)$ and describe the Specht module S^λ – that is, describe a basis and how S_n acts on it. Have you seen this representation before (possibly with another name)?

(6) (a) For a nonempty partition λ, prove that the skew Schur function $s_{\lambda/1}$ equals the sum of Schur functions s_μ over all partitions μ obtained from λ by removing a corner box. Hint: there is a very short (four-line-ish) proof!
(b) for a partition λ whose Young diagram has at least two rows and at least two columns, prove that $s_{\lambda/(2)} - s_{\lambda/(1^2)}$ equals $\sum s_\mu - \sum s_\nu$ over all partitions μ obtained from λ by removing a horizontal domino and all partitions ν obtained from λ by removing a vertical domino. Hint: use the fact that $\omega(s_\lambda) = s_{\lambda'}$, where λ' is the conjugate of λ.
(c) Find all partitions λ such that $s_{\lambda/(2)} = s_{\lambda/(1^2)}$.