(1) Show that the two definitions of matroid that we saw in class – one in terms of independent sets and one in terms of circuits – are cryptomorphic. In other words, show that if \( M \) is a matroid according to the independent set definition, then its circuits satisfy the requirements given by the circuit definition of matroid. (And vice-versa.)

(2) Let \( \mathcal{A} \) be an arrangement in \( \mathbb{R}^n \) with equations
\[
x_1 = x_2, x_2 = x_3, \ldots, x_{n-1} = x_n, x_n = x_1.
\]
Compute the characteristic polynomial \( \chi_{\mathcal{A}}(t) \) and compute the number \( r(\mathcal{A}) \) of regions of \( \mathcal{A} \).

(3) Let \( P_n \) be the permutohedron, i.e. the convex hull of all points \( \pi = (\pi_1, \pi_2, \ldots, \pi_n) \) in \( \mathbb{R}^n \) where \( \pi \) ranges over all permutations in \( S_n \). For any chain \( \emptyset \subset A_1 \subset A_2 \subset \cdots \subset A_k \subset [n] \), let \( F(A_1, \ldots, A_k) \) be the convex hull of all vertices \( \pi^{-1} \in P_n \) for permutations \( \pi \) which first list the set \( A_1 \) in some order, then \( A_2 \setminus A_1 \), then \( A_3 \setminus A_2 \), and so on.

- Show that \( F(A_1, \ldots, A_k) \) is an \((n-k-1)\)-dimensional face of \( P_n \), and that every face of \( P_n \) has this form.
- What is the number of facets of \( P_n \)? Edges?

(4) For each permutation \( \pi \in S_n \), associate the corresponding permutation matrix \( X^\pi \). Identify \( \mathbb{R}^d \) with the set of all real \( d \times d \) matrices, so the \( X^\pi \) are \( 0-1 \) vectors in \( \mathbb{R}^d \). Their convex hull forms a polytope \( B(d) \), known as the Birkhoff polytope or perfect matching polytope. Show that
- \( B(d) \) has \( d^2 \) facets and dimension \((d-1)^2\).
- \( B(d) \) is the subset of \( \mathbb{R}^{d^2} \) cut out by the inequalities \( x_{ij} \geq 0 \) for \( 1 \leq i, j \leq d \), \( \sum_{k=1}^d x_{ik} = 1 \) for \( 1 \leq i \leq d \), \( \sum_{k=1}^d x_{kj} = 1 \) for \( 1 \leq j \leq d \).