MATH 249 PROBLEM SET 4 (DUE APRIL 1)

(1) Show that the two definitions of matroid that we saw in class – one in terms of independent sets and one in terms of circuits – are *cryptomorphic*. In other words, show that if M is a matroid according to the independent set definition, then its circuits satisfy the requirements given by the circuit definition of matroid. (And vice-versa.)

(2) Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$ with equations

$$x_1 = x_2, x_2 = x_3, \ldots, x_{n-1} = x_n, x_n = x_1.$$

Compute the characteristic polynomial $\chi_{\mathcal{A}}(t)$ and compute the number $r(\mathcal{A})$ of regions of $\mathcal{A}$.

(3) Let P_n be the permutohedron, i.e. the convex hull of all points $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ in $\mathbb{R}^n$ where π ranges over all permutations in S_n. For any chain $\emptyset \subset A_1 \subset A_2 \subset \cdots \subset A_k \subset [n]$, let $F(A_1, \ldots, A_k)$ be the convex hull of all vertices $\pi^{-1} \in P_n$ for permutations π which first list the set A_1 in some order, then $A_2 \setminus A_1$, then $A_3 \setminus A_2$, and so on.
- Show that $F(A_1, \ldots, A_k)$ is an $(n-k-1)$-dimensional face of P_n, and that every face of P_n has this form.
- What is the number of facets of P_n? Edges?

(4) For each permutation $\pi \in S_n$, associate the corresponding permutation matrix X^π. Identify $\mathbb{R}^{d^2}$ with the set of all real $d \times d$ matrices, so the X^π are $0-1$ vectors in $\mathbb{R}^{d^2}$. Their convex hull forms a polytope $B(d)$, known as the *Birkhoff polytope* or *perfect matching polytope*. Show that
- $B(d)$ has d^2 facets and dimension $(d-1)^2$.
- $B(d)$ is the subset of R^{d^2} cut out by the inequalities $x_{ij} \geq 0$ for $1 \leq i, j \leq d$, $\sum_{k=1}^{d} x_{ik} = 1$ for $1 \leq i \leq d$, $\sum_{k=1}^{d} x_{kj} = 1$ for $1 \leq j \leq d$.