MATH 249 PROBLEM SET 3 (DUE MARCH 11)

(1) Let P be a locally finite poset. Define $\eta \in I(P)$ (the incidence algebra) by $\eta(x,y) = 1$ if y covers x and $\eta(x,y) = 0$ otherwise. Show that $(1-\eta)^{-1}(x,y)$ is equal to the total number of maximal chains in $[x,y]$.

(2) Recall that for any positive integer n, the partition lattice Π_n is the poset of all partitions of $[n]$ (into blocks), where we define $\pi \leq \sigma$ in Π_n if and only if each block of π is contained in a block of σ. (In other words, π is a *refinement* of σ.) Find an EL-labeling of Π_n (and prove that it is one). Then identify the homotopy-type of the order complex $\Delta(\Pi_n)$.

(3) Suppose that a permutation $\pi \in S_n$ has m inversions. Show that π can be written as a product of m simple transpositions $s_{i_1} \dots s_{i_m}$. Also, express the set of inversions of π in terms of $s_{i_1}, \dots, s_{i_m}$.

(4) Consider Young's lattice (the lattice of all partitions, ordered by containment). Calculate its Mobius function. That is, for each pair of partitions, $\lambda \subset \nu$, calculate $\mu(\lambda, \nu)$.

Note: Don't confuse Π_n with Young's lattice! Π_n is the poset of objects $(B_1, \dots, B_k)$, where the disjoint union of the B_i's is $\{1, \dots, n\}$. On the other hand, Young's lattice is the poset of all partitions $(\lambda_1, \dots, \lambda_k)$, where $\lambda_1 \geq \dots \geq \lambda_k$. We typically view this kind of partition as a Young diagram.