MATH 249 PROBLEM SET 2 (DUE FEBRUARY 25)

(1) What will be the topic of your final project?

(2) Use the Gessel-Viennot Lemma to prove the formula which expands $\det M$ according to the ith row.

(3) Prove that $\det H_n^{(1)} = 1$ for $H_n^{(1)} = (C_{i+j+1})_{i,j=0}^n$. (Use the G-V Lemma.) Here C_n denotes the nth Catalan number.

(4) Deduce from the Stirling number recurrence $S_{n,k} = S_{n-1,k-1} + kS_{n-1,k}$ the formula
$$\det(S_{m+i,j})_{i,j=1}^n = (n!)^m,$$
where $m \geq 0, n \geq 1$. Hint: use the G-V Lemma, and a graph mirroring the recurrence.

(5) Let D be a digraph with p vertices $\{v_1, v_2, \ldots, v_p\}$, and let ℓ be a fixed positive integer. Suppose that for every pair u, v of (not necessarily distinct) vertices of D, there is a unique (directed) walk of length ℓ from u to v.
  (a) Recall that the *adjacency matrix* $A(D)$ of D is the $p \times p$ matrix whose entry a_{ij} is equal to the number of edges from v_i to v_j. What are the eigenvalues of the (directed) adjacency matrix $A := A(D)$?
  (b) How many loops (v, v) does D have?
  (c) Show that every vertex has outdegree $p^{1/\ell}$.
  (d) Show that D is connected and balanced. *Hint*: You can solve explicitly for one of the eigenvectors of $A(D)$.
  (e) How many Eulerian tours does D have starting with a given edge e?