MATH 249 PROBLEM SET 1 (DUE FEBRUARY 11)

(1) A *partition* of a set S is a collection of nonempty, pairwise disjoint sets whose union is S. The sets into which S is partitioned are called the *classes* of the partition. So for example, there are 7 partitions of $\{1, 2, 3, 4\}$ into two classes. Let $S(n, k)$ be the number of partitions of $\{1, 2, \ldots, n\}$ into k classes. This is called the Stirling number of the second kind. By convention, we say that $S(n, k) = 0$ if $k > n$ or $n < 0$ or $k < 0$. We will also say that $S(n, 0) = 0$ if $n \neq 0$ and $S(0, 0) = 1$.

(a) Find a recurrence for the Stirling numbers.
(b) Let $B_k(x) = \sum_{n\geq 0} S(n, k)x^n$. Find an explicit expression for $B_k(x)$.
(c) Use partial fractions to show that $S(n, k) = \sum_{r=1}^{k} (-1)^{k-r} \frac{r^n}{r!(k-r)!}$ for $n, k \geq 0$.

(2) Prove that the Prufer code is indeed a bijection between rooted trees on vertices $\{1, 2, \ldots, n\}$ and words of length $n - 1$ on the alphabet $\{1, 2, \ldots, n\}$.

(3) Let $c_{n,i,k}$ be the number of graphs on n vertices, with i edges and k components. Show that

$$\sum_{n,k,i\geq 0} c_{n,i,k}\alpha^i\beta^k \frac{z^n}{n!} = \left(\sum_{n\geq 0} (1 + \alpha)^{\binom{n}{2}} \frac{z^n}{n!}\right)^{\beta}.$$

(4) Fix some positive number $k \geq 2$. Show that the number of partitions of n in which every part appears at most $k - 1$ times equals the number of partitions where every part is not divisible by k.