(1) Let $M = (E, \mathcal{B})$ be a matroid and $S \subset E$. Show that

$$(M/S)^* = M^*|_{E-S}. $$

(2) Let $f$ be a function from matroids to $\mathbb{C}$ such that:

(a) If $E = \emptyset$ then $f(M) = 1$

(b) Let $A = f(\text{coloop})$ and let $B = f(\text{loop})$. Then

$$f(M) = Af(M - e) \text{ for } e \text{ a coloop, and}$$
$$f(M) = Bf(M - e) \text{ for } e \text{ a loop.}$$

(c) There exist constants $\alpha, \beta \neq 0$ such that when $e$ is not a loop or coloop, we have

$$f(M) = \alpha f(M - e) + \beta f(M/e).$$

Show that for all matroids $M$,

$$f(M) = \alpha^{\lvert E \rvert - r(E)}\beta^{r(E)}T_M(A/\beta, B/\alpha),$$

where $T_M$ is the Tutte polynomial.

(3) Recall that the chromatic polynomial is the polynomial enumerating proper colorings of a graph. Show that the chromatic polynomial satisfies a deletion-contraction recurrence. Then deduce that

$$\chi_G(\lambda) = (-1)^{|V| - k(G)}\lambda^{k(G)}T_G(1 - \lambda, 0),$$

where $k(G)$ equals the number of connected components of the graph.

(4) How many positroids of rank 2 on the ground set $[4]$ are there? Hint: it might help to use the three-term Plücker relation.