**Math 113 Homework 8, due 3/22/2012 at the beginning of section**

*Policy: if you worked with other people on this assignment, write their names on the front of your homework. Remember that you must write up your solutions independently.*

**1.** Fraleigh section 16, exercises 8, 13.

**2.** Fraleigh section 17, exercises 4, 5.

**3.** (Note: in this exercise, please don't call the additive and multiplicative identity elements 0 and 1 if there is any risk of confusion.)

(a) Let $F$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$, equipped with the operations of function addition $(f+g)(x) = f(x)+g(x)$ and composition $(f \circ g)(x) = f(g(x))$. Show that $(F, +, \circ)$ satisfies all the axioms of a ring with unity, with just one exception – which one?

(b) Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be the set formed by adjoining an element called "$\infty$" to $\mathbb{R}$, and consider the operations $a \oplus b = \min(a, b)$ (= the lesser of $a$ and $b$; with the convention that $\infty$ is greater than any real number), and $a \otimes b = a + b$ (with the convention that $a + \infty = \infty + a = \infty$ for all $a \in \mathbb{R}$, and $\infty + \infty = \infty$). Show that $(\bar{\mathbb{R}}, \oplus, \otimes)$ satisfies all the axioms of a field, with just one exception – which one?

(c) Show that $\{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$ with the usual addition and multiplication is a field.

**4.** True or false? (As usual, justify your answers) For part (h), you'll need to know that an *integral domain* is a commutative ring with unity $1 \neq 0$ such that there are no elements $a, b$ such that $ab = 0$.

(a) The set of all pure imaginary complex numbers $\{ai \mid a \in \mathbb{R}\}$ with the usual addition and multiplication is a ring.

(b) If $R'$ is a subring of a field $K$, then $R'$ is also a field.

(c) If $K$ is a field then the equation $x^2 = x$ has exactly two solutions in $K$.

(d) If $K$ is a field with unity 1 and $K'$ is a subfield of $K$ with unity $1'$, then $1' = 1$. (Hint: use (c)).

(e) If $R$ is a ring with unity $1 \neq 0$ and $R'$ is a subring with unity $1' \neq 0$, then $1' = 1$. (Hint: consider a direct product.)

(f) The set of $GL(n, \mathbb{R})$ of invertible $n \times n$ matrices with entries in $\mathbb{R}$, with matrix addition and multiplication, is a skew field.

(g) The direct product of two fields is a field.

(h) The nonzero elements of an integral domain form a group under multiplication.

(i) A divisor of zero in a commutative ring with unity cannot have a multiplicative inverse.

**5.** How challenging did you find this assignment? How long did it take?