Math 113 Homework 4, due 2/16/2012 at the beginning of section

Policy: if you worked with other people on this assignment, write their names on the front of your homework. Remember that you must write up your solutions independently.

1. Fraleigh, section 8, problem 21; section 9, problem 23.

2. (a) Exhibit an element of S_9 which has order 20.

(b) Prove that S_9 has no element of order 18.

(c) A perfect shuffle of a deck of cards can be represented by the following permutation $f \in S_{52}$:

$$f(x) = \begin{cases} 2x-1, & \text{if } x \in \{1, \dots, 26\}, \\ 2(x-26), & \text{if } x \in \{27, \dots, 52\}. \end{cases}$$

Show that, if one performs 8 perfect shuffles of a deck of cards, then this returns the cards to their original position. (Suggestion: first decompose f into a product of disjoint cycles.)

3. (a) Show that if $\mu = (x_1\ x_2 \cdots x_k) \in S_n$ is a k-cycle and $\sigma \in S_n$ is any permutation, then $\sigma\mu\sigma^{-1}$ is the k-cycle $\sigma\mu\sigma^{-1} = (\sigma(x_1)\ \sigma(x_2) \cdots \sigma(x_k))$.

(b) Using the above, find a necessary and sufficient condition for two permutations in S_n to be conjugate to each other. (As usual, justify your answer). (Recall: $x, y \in S_n$ are conjugate if there exists $\sigma \in S_n$ such that $y = \sigma x \sigma^{-1}$).

4. Show that, if $n \geq 3$, then the center of S_n is the trivial subgroup $\{1\}$. (see Homework 3 for the definition of the center; use the results of the previous problem if needed).

5. This problem will be due Thursday Feb 23 as part of homework 5.

Let G be the symmetry group of a cube (with no reflections allowed), i.e. the group of space rotations which preserve a cube. Show that $G \simeq S_4$.

Hint: a cube has four diagonals which connect opposite vertices and go through the center of the cube. Any element of G induces a permutation of the set of diagonals. It is enough to show that every permutation of the set of diagonals is realized by some element of G. (Why? What is the order of G?)

(If it helps, you can use without proof the fact that an isometry of space which fixes the origin and is represented by a matrix of determinant $+1$ is a rotation about some axis).

6. How challenging did you find this assignment? How long did it take?