Math 113: Quotient Group Computations

Fraleigh's book doesn't do the best of jobs at explaining how to compute quotient groups of finitely generated abelian groups. I'd say the most useful example from the book on this matter is Example 15.11, which involves the quotient of a *finite* group, but does utilize the idea that one can figure out the group by considering the orders of its elements. This idea will take us quite far if we are considering quotients of finite abelian groups or, say, quotients $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}/\langle x \rangle$ where $\langle x \rangle$ is a cyclic subgroup. This idea of considering orders of elements is one that you should be fairly comfortable with. In these notes, we begin by working out several examples by applying this idea alone. We then move on to a more algorithmic method of computing such quotients (and much more difficult ones) that might be a bit foreign to students who haven't had much experience with linear algebra. Basically, this second method enables one to find, given a finitely generated abelian group G and a normal subgroup H (all subgroups of G are normal and finitely generated), a surjective homomorphism $\phi : G \to G'$ such that the kernel $ker(\phi) = H$. As we know well, this implies that $G/H \cong G'$. From reading these notes, you can decide which method you like better. I prefer the first in light of what you have learned in this class.

The Orders of Elements Approach:

The basic idea here is always the same:

- Determine the cosets of the subgroup
- Narrow down which finitely generated abelian group the quotient can be
- Decide which one exactly it is by considering orders of cosets

Let's see how this works in three different examples.

Example 1: $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/\langle(1,1,2)\rangle \cong \mathbb{Z} \times \mathbb{Z}$.

Let's denote by H the group $\langle(1,1,2)\rangle$ and find all of the distinct cosets of H. In fact, the only cosets are of the form

$$(m,n,0)H \text{ and } (m,n,1)H$$

where m and n are any integers. I leave it to you to show this is true. The way to see such cosets are distinct is to show that their difference is not contained in H. Then, to show that these are the only cosets you can indicate how one would get from an arbitrary coset $(a,b,c)H$ to one of these by subtracting off some element of H from (a,b,c). Now, we see that this group contains $\mathbb{Z}\times\mathbb{Z}$, but it's definitely not $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$, so it might be $\mathbb{Z}\times\mathbb{Z}$, or $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_k$ for some k. To figure out which of these we're looking at, let's try to solve the equation

$$(mi,ni,xi) = (j,j,2j)$$

for some integer $i > 0$ and some integer j. This is the same as asking what the elements of finite order are. We have $x = 0$ or $x = 1$ in our cosets. In the first case, we're solving

$$(mi,ni,0i) = (j,j,2j)$$

and we see that $j = 0$. Then $m = n = 0$, so the only element of finite order with a 0 in the third coordinate is $(0,0,0)H$, the trivial coset.

Now, let's try $x = 1$. Then we're solving

$$(mi,ni,i) = (j,j,2j)$$

so $i = 2j$. But then $2mj = 2nj = j$ so $m = n = 1/2$ but our elements only have integers appearing in the coordinates, so there are no elements of finite order with a 1 in the last coordinate in our group. That exhausts all the elements possible, so we have found that just the trivial element has finite order. Therefore our group is $\mathbb{Z}\times\mathbb{Z}$.

The next two examples are the two problems you had on Homework 6.

Example 2: $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/\langle(3,3,3)\rangle \cong \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_3$.

To see this, first note that there are precisely three different kinds of cosets which are all different from one another: $(m,n,0)H$ where m and n are any integers, $(m,n,1)H$ where m and n are any integers, and $(m,n,2)H$ where m and n are any integers and H denotes the group $\langle(3,3,3)\rangle$ (I leave it up to you to check that these are precisely all the different cosets one can get, using the same strategy as the one outlined above). Note that our quotient group obviously contains a subgroup isomorphic to $\mathbb{Z}\times\mathbb{Z}$ which can be seen by considering just the cosets of the form $(m,n,0)$. So, since this quotient group is a finitely generated abelian group which contains $\mathbb{Z}\times\mathbb{Z}$, and since it certainly can't contain $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ as a proper subgroup, the options are

- $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$
- $\mathbb{Z}\times\mathbb{Z}$
- $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_{p_1^{r_1}}\times\mathbb{Z}_{p_2^{r_2}}\times\cdots\times\mathbb{Z}_{p_n^{r_n}}$

Note that in the first two options there are no non-identity elements of finite order, and in the last option there are elements of order $p_i^{r_i}$ for every i. So, let's see whether our quotient group has elements of finite order. We check for this as follows. Let $(m,n,x)H$ be an arbitrary coset in the group, where $x=0,1$, or 2. We have to consider what possible positive integer solutions i,j there are to

$$(mi,ni,xi)=(3j,3j,3j).$$

If $x=0$, we have $0i=0=3j$ so i can be anything and j must be 0. This gives us $(mi,ni,0)=(0,0,0)$, and thus $m,n=0$. So the only finite order coset with $x=0$ is $(0,0,0)H$, the identity coset. Now, if $x=1$, we have $1i=i=3j$, and so $(mi,ni)=(3mj,3mj)=(3j,3j)$ and so $m,n=1$ and the smallest positive value for j is 1. This means the only finite order coset with $x=0$ is $(1,1,1)H$ and it is of order 3. Similarly, we can show that the only finite order coset with $x=2$ is $(2,2,2)H$ and it also order 3. So our quotient group has 3 elements of finite order: one of order 1 and two of order 3. Therefore the first two options above won't do, and the only group which fits into the third option and has just two non-identity finite order elements both of order 3 is $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_3$, as claimed.

Example 3: $\mathbb{Z}\times\mathbb{Z}/\langle(6,9)\rangle\cong\mathbb{Z}\times\mathbb{Z}_3$. We will proceed as before. There is more than one way we can describe the cosets of $H=\langle(6,9)\rangle$ in $\mathbb{Z}\times\mathbb{Z}$, but here is one possible classification of all the cosets:

- $(0,n)H$ where n is any integer,
- $(1,n)H$ where n is any integer,

- $(2,n)H$ where n is any integer,
- $(3,n)H$ where n is any integer,
- $(4,n)H$ where n is any integer,
- $(5,n)H$ where n is any integer.

Again, I leave it to you to show that these are all absolutely distinct cosets, and that any other coset is equal to one of these (note that $(a,b)H = (6k+r,b)H = (r,b-9k)H$ for any integers a and b, where $r = 0,1,2,3,4,$ or 5 as above). Also, it is easy to see that this quotient group contains a subgroup isomorphic to $\mathbb{Z}$: namely, consider the cosets $(0,n)H$ which form a subgroup of the group of all cosets, and this is visibly isomorphic to $\mathbb{Z}$. So, according to the fundamental theorem of finitely generated abelian groups, our group might be one of the following:

- $\mathbb{Z}$
- $\mathbb{Z}\times\mathbb{Z}$
- $\mathbb{Z}\times\mathbb{Z}_{p_1^{r_1}}\times\mathbb{Z}_{p_2^{r_2}}\times\cdots\times\mathbb{Z}_{p_n^{r_n}}$

Note as in the last example that in the first two options there are no non-identity elements of finite order, and in the last option there are elements of order $p_i^{r_i}$ for every i. Also, note that our quotient group does have non-identity elements of finite order: take for example the coset $(2,3)H$. This rules out the first two options. Now let's find all the cosets of finite order here. Let $(x,n)H$ be an arbitrary coset in the group, where $x = 0,1,2,3,4$ or 5. We have to consider what possible positive integer solutions i, j there are to

$$(xi,ni) = (6j,9j).$$

If $x = 0$, we have $0i = 0 = 6j$ so i can be anything and j must be 0. This gives us $(0,ni) = (0,0)$, and thus $n = 0$. So the only finite order coset with $x = 0$ is $(0,0)H$, the identity coset. Now, if $x = 1$, we have $1i = i = 6j$, and so $ni = 6jn = 9j$ and so $n = 9/6 = 3/2$ but this is not an integer, so there are no finite order cosets with $x = 1$. Similarly, we can show that there are no finite order cosets with $x = 3$ or $x = 5$. However, if $x = 2$ we have $2i = 6j$, so $i = 3j$, and so $ni = 3jn = 9j$ so $n = 3$. So the only finite order coset with $x = 2$ is $(2,3)H$, of order 3. Similarly, if $x = 4$, then $4i = 6j$ and

$i = 6j/4 = 3j/2$ and so $ni = 3nj/2 = 9j$ meaning $n = 6$. So the only finite order coset with $x = 4$ is $(4,6)H$, of order 3. This exhausts all the possible finite order cosets, and so our quotient group has 3 elements of finite order: one of order 1 and two of order 3. So, as in the last example, our group has just two non-identity finite order elements both of order 3 and must be $\mathbb{Z} \times \mathbb{Z}_3$, as claimed.

The Linear Algebra-esque Approach:

Another way to show that $G/N \cong G'$ is to cook up a surjective homomorphism $\phi : G \to G'$ whose kernel is N. Doing this using trial and error can often be tedious and unsuccessful, but in fact there is a very neat method which always gives you the answer, at least in the kinds of examples we've studied so far. A word of warning: it isn't hard to blindly apply this algorithm without understanding why it works. If you don't understand the method, it's probably not such a good idea to use it, and it's better to use the method outlined above. However, if you do use it, at least be sure to check that the homomorphism you get really does do the job. For example, if I were grading an exam problem in which a student applies this method incorrectly, and doesn't seem at all aware that their answer is wrong, I would probably not give much credit. To check if your answer is right, calculate what the kernel of the homomorphism is. It should be the normal subgroup N.

To explain how to use "linear algebra" (it's not exactly linear algebra, but looks like it), we have to be comfortable with groups generated by several elements: in this class we have mostly studied groups generated by one element. Recall that a subgroup of a group G is generated by elements $g_1, g_2, g_3, \dots g_k \in G$ if and only if it is the smallest group containing all of these elements. This is equivalent to saying that it is generated by all possible "products" of these elements and their inverses (I put products in quotes since we're not actually multiplying the elements but applying whatever binary operation $*$ the group G comes with). So, when we looked at cyclic groups we were just looking at groups generated by one element a, and the group then contained the identity and all elements of the form $a * a * a * \cdots * a$ or $a^{-1} * a^{-1} * a^{-1} * \cdots * a^{-1}$. A group generated by $g_1, g_2, g_3, \dots g_k$ consists of the identity and all elements of the form $h_{a_1} * h_{a_2} * \cdots * h_{a_m}$ where h_{a_i} can be any of the generators g_j or one of the generators' inverses.

That said, let's denote the group $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ with n factors of $\mathbb{Z}$ by $\mathbb{Z}^n$. Note that this group is generated by $e_1 = (1,0,0,\dots 0), e_2 = (0,1,0,\dots,0), e_3 = (0,0,1,0,\dots 0), \dots,$ and $e_n = (0,0,\dots,0,1)$. We denote these by e_i since this is how one usually refers to this basis of $\mathbb{Z}^n$, or

$\mathbb{R}^n$, for that matter. We will henceforth refer to these as "basis vectors". By the way, this group is not only finitely generated and abelian, it is also what's called "free," which for the purpose of finitely generated abelian groups just means there are no finite factors in the decomposition into cyclic groups. The method we'll outline can be used to handle the non-free cases as well, but we'll leave that case alone for the purpose of these notes.

Now, suppose we are trying to classify group

$$\mathbb{Z}^n / \langle (a_{11}, a_{12}, a_{13}, \ldots, a_{1n}), (a_{21}, a_{22}, \ldots, a_{2n}), \ldots (a_{k1}, a_{k2}, \ldots, a_{kn}) \rangle$$

according to the fundamental theorem of finitely generated abelian groups. We do this is by changing the basis or generators of $\mathbb{Z}^n$ together with the basis or generators of the group we are factoring by to make it more convenient to work with. There are two tricks we will use, essentially: one is changing the basis $e_1, \ldots, e_n$ which will also have an effect on the presentation of the group we are factoring by. Another is to change the generators of the group we're factoring by which won't have an effect on the basis vectors $e_1, \ldots e_n$. Note first of all that the ith generator of the group $H = \langle (a_{11}, a_{12}, a_{13}, \ldots, a_{1n}), (a_{21}, a_{22}, \ldots, a_{2n}), \ldots (a_{k1}, a_{k2}, \ldots, a_{kn}) \rangle$ can be written as

$$h_i = \sum_{j=1}^{n} a_{ij} e_j.$$

Therefore, whenever we exchange the basis vectors e_i and e_j we change the generators of H (without changing H itself!) by exchanging the ith and jth coordinate of each generator. Multiplying a basis vector e_i by -1 induces multiplication of the ith coordinate of each generator of H by -1. Finally, note that replacing e_i by $e_i + me_j$ for some integer m makes the following change to each generator of H: it *subtracts* m times the ith coordinate from the jth coordinate. This might seem confusing, but consider the following example: say we have the basis e_1, e_2 which we replace by $e_1 + 2e_2, e_2$. Then the pair $(5,6) = 5e_1 + 6e_2 = 5(e_1 + 2e_2) + (6 - 5 \cdot 2)e_2 = 5(e_1 + 2e_2) - 4e_2$. Therefore in these new coordinates $(5,6)$ has been switched to $(5,-4)$. This explains how changing the basis vectors e_i will affect the generators of H. Now, as we mentioned above, the other trick we have up our sleeve is to change the generators of H: for example, we can change the ith generator h_i to $h_i + mh_j$ where m is an integer and h_j is another generator. Note this won't change the group at all.

Let's combine this information as follows. Make up a $k \times n$ matrix M whose rows are the generators of H. Just like we did in linear algebra, we can perform some elementary row and column operations on this matrix. We summarize what these are and what these correspond to as far as our groups go in the following table, where the first column lists operations on M, and the second column lists the corresponding effects on the basis vectors e_i and the genrators h_i of H (note that m denotes an integer below).

Switch the ith and jth rows of M	Switch the generators h_i and h_j of H
Switch ith and jth columns of M	Switch e_i and e_j
Multiply the ith row by -1	Replace the generator h_i by $-h_i$
Multiply the ith column by -1	Replace e_i with $-e_i$
Add m times the ith row to the jth row of M	Replace the generator h_j with $h_j + mh_i$
Add m times the ith column to the jth column of M	Replace e_i with $e_i - me_j$

You might ask, why are we only allowed to multiply by ± 1 the rows and columns of M in the third and fourth lines of the table above? This is because multiplying by anything else actually changes the group H or the group $\mathbb{Z}^n$ we started with. For example, the subgroup of $\mathbb{Z}^3$ generated by $(3,4,5)$ is the same as the subgroup generated by $(-3,-4,-5)$, but is different from the group generated by $(6,8,10)$.

In any case, if we perform these operations over and over what we can arrive at is a matrix M' which is now diagonal and consists only of non-negative entries, meaning $M'_{ij} = 0$ whenever $i \neq j$, and $M'_{ij} \geq 0$ for all i, j. We omit the proof that arriving at a diagonal matrix is indeed always possible from these notes, but it can be shown by repeated application of the Euclidean algorithm. Below are some examples of "diagonal matrices" which are not necessarily square matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 8 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 3 & 0 \end{pmatrix}, \begin{pmatrix} 8 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

While doing this, we have changed the generators of H: they are now of the form $h'_1 = (x_1, 0, \dots, 0), h'_2 = (0, x_2, 0, \dots, 0), \dots, h'_k = (0, \dots, x_k, 0, \dots, 0)$. We have also changed the basis vectors somewhat, which just means that we think of elements of $\mathbb{Z}^n$ as linear combinations of $e'_1, e'_2, \dots, e'_n$, where e'_i

are our new basis vectors. Now what happens is that we can easily define an onto homomorphism

$$\begin{aligned}\phi : \mathbb{Z}^n &\to \mathbb{Z}_{x_1} \times \mathbb{Z}_{x_2} \times \cdots \times \mathbb{Z}_{x_k} \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \\ \phi(e_i') &= e_i' \text{ for every } 1 \le i \le n.\end{aligned}$$

where there are $n-k$ factors of $\mathbb{Z}$ at the end in the image. Essentially, working in this new basis, ϕ reduces the ith coordinate mod x_i for $1 \le i \le k$, and leaves all the other coordinates alone. This defines a homomorphism between the groups since any element of $\mathbb{Z}^n$ is a linear combination of the vectors e_i'. Its kernel, furthermore, consists of elements $(b_1, b_2, \ldots, b_n)$ where $b_i \equiv 0 \pmod{x_i}$ for $1 \le i \le k$ and $b_i = 0$ for $k < i \le n$. Note that in our new basis $e_1', \ldots e_n'$ this is obviously the group H, and so in our original basis $e_1, \ldots, e_n$ this kernel is the group H as well. We will see this in a few examples below. So we have found a homomorphism from $\mathbb{Z}^n$ to $\mathbb{Z}_{x_1} \times \mathbb{Z}_{x_2} \times \cdots \times \mathbb{Z}_{x_k} \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ with kernel H, meaning that $\mathbb{Z}^n/H \cong \mathbb{Z}_{x_1} \times \mathbb{Z}_{x_2} \times \cdots \times \mathbb{Z}_{x_k} \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$.

* Note that $\mathbb{Z}_1$ is just the trivial group, so if you'd like you can just throw any factor $\mathbb{Z}_{x_i} = \mathbb{Z}_1$ out at the end.

* Note that if one of the diagonal entries of M' (i.e. in an iith spot) is 0, $\mathbb{Z}_0$ can/should be replaced by $\mathbb{Z}$.

Let's see this method in action for the examples described at the beginning of these notes, as well as for a couple other examples where the method described in the beginning will be sure to give you a major headache.

Example 4: $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/\langle(1,1,2)\rangle \cong \mathbb{Z} \times \mathbb{Z}$.

Here the matrix M which we have to diagonalize using the row and column operations outlined above is just $M = (1\ \ 1\ \ 2)$. We do this as follows.

$$(1\ \ 1\ \ 2) \longrightarrow (1\ \ 0\ \ 2) \longrightarrow (1\ \ 0\ \ 0)$$

The first step was to add -1 times the first column to the second column; then add -2 times the first column to the third column. So now the group we're factoring by is generated by $(1,0,0)$ in which basis? To check this, we use our table above to translate our column operations into the following change of basis. First $(e_1,e_2,e_3) \to (e_1+e_2,e_2,e_3)$, then $(e_1+e_2,e_2,e_3) \to (e_1+e_2+2e_3,e_2,e_3)$. So, the homomorphism ϕ described above is

$$\phi : \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}_1\times\mathbb{Z}\times\mathbb{Z} \cong \mathbb{Z}\times\mathbb{Z}$$

and $\phi(e_1') = \phi(e_1+e_2+2e_3) = (1,0,0)$, $\phi(e_2') = \phi(e_2) = (0,1,0)$, $\phi(e_3') = \phi(e_3) = (0,0,1)$. Using the fact that ϕ is a homomorphism, this tells us that $\phi(e_1) = (1,-1,-2)$. So we have

$$\phi(a,b,c) = \phi(a\cdot e_1 + b\cdot e_2 + c\cdot e_3) = (a, b-a, c-2a).$$

By our discussion above, the image of ϕ is $\mathbb{Z}\times\mathbb{Z}$, and the kernel is easily seen to be $\langle(1,1,2)\rangle$, as desired. Thus $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/\langle(1,1,2)\rangle \cong \mathbb{Z}\times\mathbb{Z}$.

Example 5: $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/\langle(3,3,3)\rangle \cong \mathbb{Z}_3\times\mathbb{Z}\times\mathbb{Z}$.

Here the matrix M which we have to diagonalize using the row and column operations outlined above is just $M = (3\ \ 3\ \ 3)$. We do this as follows.

$$(3\ \ 3\ \ 3) \longrightarrow (3\ \ 0\ \ 3) \longrightarrow (3\ \ 0\ \ 0)$$

The first step was to add -1 times the first column to the second column; then add -1 times the first column to the third column. So now the group we're factoring by is generated by $(1,0,0)$ in which basis? To check this, we use our table above to translate our column operations into the following change of basis. First $(e_1,e_2,e_3) \to (e_1+e_2,e_2,e_3)$, then $(e_1+e_2,e_2,e_3) \to (e_1+e_2+e_3,e_2,e_3)$. So, the homomorphism ϕ described above is

$$\phi : \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}_3\times\mathbb{Z}\times\mathbb{Z}$$

and $\phi(e_1') = \phi(e_1+e_2+e_3) = (1,0,0)$, $\phi(e_2') = \phi(e_2) = (0,1,0)$, $\phi(e_3') = \phi(e_3) = (0,0,1)$. Using the fact that ϕ is a homomorphism, this tells us that $\phi(e_1) = (1,-1,-1)$. So we have

$$\phi(a,b,c) = \phi(a\cdot e_1 + b\cdot e_2 + c\cdot e_3) = (a, b-a, c-a).$$

By our discussion above, the image of ϕ is $\mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}$, and the kernel is easily seen to be $\langle(3,3,3)\rangle$, as desired. Thus $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/\langle(3,3,3)\rangle \cong \mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}$.

Example 6: $\mathbb{Z} \times \mathbb{Z}/\langle(6,9)\rangle \cong \mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}$.

Here the matrix M which we have to diagonalize using the row and column operations outlined above is just $M = (6\ \ 9)$. We do this as follows.

$$(6\ \ 9) \longrightarrow (6\ \ 3) \longrightarrow (3\ \ 3) \longrightarrow (3\ \ 0)$$

The first step was to add -1 times the first column to the second column; then add -1 times the second column to the first column; and finally add -1 times the first column to the second column. So now the group we're factoring by is generated by $(3,0)$ in which basis? To check this, we use our table above to translate our column operations into the following change of basis. First $(e_1,e_2) \to (e_1+e_2,e_2)$, then $(e_1+e_2,e_2) \to (e_1+e_2,e_1+2e_2)$, and finally $(e_1+e_2,e_1+2e_2) \to (2e_1+3e_2,e_1+2e_2)$. So, the homomorphism ϕ described above is

$$\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_3 \times \mathbb{Z}$$

where $\phi(e_1') = \phi(2e_1+3e_2) = (1,0)$ and $\phi(e_2') = \phi(e_1+2e_2) = (0,1)$. Using the fact that ϕ is a homomorphism, this tells us that $\phi(e_1) = \phi(2e_1' - 3e_2') = (2,-3)$ and $\phi(e_2) = (-1,2)$. So we have

$$\phi(a,b) = \phi(a \cdot e_1 + b \cdot e_2) = (2a-b, 2b-3a).$$

By our discussion above, the image of ϕ is $\mathbb{Z}_3 \times \mathbb{Z}$, and the kernel can be seen to be $\langle(6,9)\rangle$, as desired. Thus $\mathbb{Z} \times \mathbb{Z}/\langle(6,9)\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}$.

Exercise: convince yourself using the Euclidean Algorithm that $\mathbb{Z} \times \mathbb{Z}/\langle(a,b)\rangle \cong \mathbb{Z}_{gcd(a,b)} \times \mathbb{Z}$.

Example 7: $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/\langle(2,3,11)\rangle \cong \mathbb{Z} \times \mathbb{Z}$.

Here the matrix M which we have to diagonalize using the row and column operations outlined above is just $M = (2\ \ 3\ \ 11)$. We do this as follows.

$$(2\ \ 3\ \ 11) \longrightarrow (2\ \ 1\ \ 11) \longrightarrow (2\ \ 1\ \ 0) \longrightarrow (0\ \ 1\ \ 0) \longrightarrow (1\ \ 0\ \ 0)$$

The first step was to add -1 times the first column to the second column; then add -11 times the second column to the third column; then add -2 times the second column to the first column; then switch the first column and the second column. So now the group we're factoring by is generated by $(1,0,0)$ in which basis? To check this, we use our table above to translate our column operations into the following change of basis. First $(e_1,e_2,e_3) \to (e_1+e_2,e_2,e_3)$, then $(e_1+e_2,e_2,e_3) \to (e_1+e_2,e_2+11e_3,e_3)$, then $(e_1+e_2,e_2+11e_3,e_3) \to (e_1+e_2,3e_2+11e_3+2e_1,e_3)$, and finally $(e_1+e_2,3e_2+11e_3+2e_1,e_3) \to (3e_2+11e_3+2e_1,e_1+e_2,e_3)$. So, the homomorphism ϕ described above is

$$\phi : \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}_1\times\mathbb{Z}\times\mathbb{Z} \cong \mathbb{Z}\times\mathbb{Z}$$

and $\phi(e_1') = \phi(2e_1+3e_2+11e_3) = (1,0,0)$, $\phi(e_2') = \phi(e_1+e_2) = (0,1,0)$, $\phi(e_3') = \phi(e_3) = (0,0,1)$. Using the fact that ϕ is a homomorphism, this tells us that $\phi(e_1) = \phi(-e_1'+3e_2'+11e_3') = (-1,3,11)$ and $\phi(e_2) = (1,-2,-11)$. So we have

$$\phi(a,b,c) = \phi(a\cdot e_1 + b\cdot e_2 + c\cdot e_3) = (-a+b, 3a-2b, 11a-11b+c).$$

By our discussion above, the image of ϕ is $\mathbb{Z}_1\times\mathbb{Z}\times\mathbb{Z}$, and the kernel is precisely $\langle(2,3,11)\rangle$, as desired (it has to be by our discussion from above). As you can see "guessing" this homomorphism would already have been a pain. But using our row-column operations method we computed $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/\langle(2,3,11)\rangle \cong \mathbb{Z}\times\mathbb{Z}$.

For the last example, let's try modding out by a non-cyclic group (i.e. by a group with more than one generator). You don't have to get comfortable with this for exam purposes, but it's a good illustration of how this method works in less friendly cases.

Example 8: $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/\langle(1,2,3),(2,3,6),(3,6,12),(6,9,15)\rangle \cong \mathbb{Z}_3$.

Here the matrix M which we have to diagonalize using the row and column operations outlined above is a 4×3 matrix, and we diagonalize it as follows.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 6 \\ 3 & 6 & 12 \\ 6 & 9 & 15 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & 0 & 3 \\ 6 & -3 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \\ 0 & -3 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

The first step was to add -2 times the first column to the second column; then add 2 times the second column to the first column; then multiply the second row by -1; and finally add 3 times the second row to the fourth row, and the third row to the fourth row (this last operation does nothing to our basis vectors). So now the group we're factoring by is generated by $(1,0,0)$, $(0,1,0)$, and $(0,0,3)$ in which basis? To check this, we use our table above to translate our column operations into the following change of basis. First $(e_1,e_2,e_3) \to (e_1+2e_2,e_2,e_3)$, then $(e_1+2e_2,e_2,e_3) \to (e_1+e_2,2e_1+3e_2,e_3)$, then $(e_1+e_2,2e_1+3e_2,e_3) \to (e_1+e_2,-2e_1-3e_2,e_3)$. So, the homomorphism ϕ described above is

$$\phi : \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}_1\times\mathbb{Z}_1\times\mathbb{Z}_3 \cong \mathbb{Z}_3$$

and $\phi(e_1') = \phi(e_1+e_2) = (1,0,0)$, $\phi(e_2') = \phi(-2e_1-3e_2) = (0,1,0)$, $\phi(e_3') = \phi(e_3) = (0,0,1)$. Using the fact that ϕ is a homomorphism, this tells us that $\phi(e_1) = \phi(3e_1'+e_2') = (3,1,0)$ and $\phi(e_2) = (-2,-1,0)$. So we have

$$\phi(a,b,c) = \phi(a\cdot e_1 + b\cdot e_2 + c\cdot e_3) = (3a-2b, a-b, c).$$

By our discussion above, the image of ϕ is $\mathbb{Z}_1\times\mathbb{Z}_1\times\mathbb{Z}_3$, and the kernel has to be

$$\langle(1,2,3),(2,3,6),(3,6,12),(6,9,15)\rangle.$$

This time it's even more doubtful that one could have guessed this homomorphism off the top of one's head.