Math 1B

Midterm 2 Review

Sequences and Series

Sequences

In this course we rarely deal with plain sequences, however it is important to distguish *sequences* from *series*. A sequence is simply a sequence (in the usual plain English sense) of numbers and they are typically labelled with a subscript n, e.g. $(a_n)_n$ or $\{a_n\}$.

We have three ways to represent series in this class. Namely,

(i) As a list, e.g.

$$1, 2, 4, 8, \ldots$$

(ii) Explicitly via a formula, e.g.

$$a_n = 2^n$$

(iii) Recursively, e.g.

$$a_n = 2a_{n-1}, \quad a_0 = 1.$$

Note that recursive definitions also require us to specify starting terms. As a rule of thumb, a recursive definition referencing the previous m terms will initial values for the first m terms in the sequence.

Exercises

1. Find the 3rd term of the following sequence

$$a_n = \frac{(-1)^{n-1}2^n}{n!}.$$

2. Find an explicit formula for the following sequence

$$\frac{3}{2}, \frac{5}{4}, \frac{7}{8}, \frac{9}{16}, \ldots$$

Hint: Look at how the numerators and denominators behave separately.

3. Find the 5th term of the following sequence

$$b_{n+1} = b_n + b_{n-1}, \quad b_0 = b_1 = 1.$$

Series

In this class, series refer to infinite sums. Just as we can list out sequences or give explicit formulas, we can do the same with series e.g.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

and equivalently

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

In general, lets say we want to add up all the terms of some sequence $\{a_n\}$, that is we want to compute

$$\sum_{n=1}^{\infty} a_n.$$

Now, it doesn't make sense to add up infinitely many things so we need to make a definition for what this means. To do this, we define *partial sums* of the series. The Nth partial sum is given by

$$\begin{aligned} S_N &= \sum_{n=1}^{N} a_n \\ &= a_1 + a_2 + \cdots + a_N \end{aligned}$$

and is the sum of the first N terms. We then make sense of the infinite sum by saying that it exists if, as we add more and more terms, the value we get approaches some number which then becomes the value of the infinite sum. That is we set

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N.$$

If this limit exits, we say that $\sum_n a_n$ is *convergent*. Otherwise, we say that $\sum_n a_n$ is *divergent*.

Exercises

1. Using the definition of infinite series via partial sums, compute

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

2. Using partial fraction decomposition and the definition of infinite series via partial sums, compute

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2(n+2)^2}.$$

3. Arguing from definition, show that

$$\sum_{n=1}^{\infty} 1$$

diverges.

Explicit Tools for Assessing Convergence

Geometric Series and p-Series

We have the following two theorems which you should remember which help us identify the convergence and divergence of series.

Theorem 1 (Geometric Series Test). *The series*

$$\sum_{n=0}^{\infty} ar^n$$

diverges if $|r| \geq 1$ *and converges if* $|r| < 1$. *In the case of* $|r| < 1$, *we have that*

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

Note that in this theorem it is important that n starts at $n = 0$, at least for the second half which asserts the value of the sequence. On exams, we will often be given a severely mangled geometry series like the following

$$\sum_{n=3}^{\infty} \frac{2^{n+1}}{3^{2n}}$$

and we will be asked to compute its value. This is bad because not only is it not clear what a and r are, but it also starts at $n = 3$. For these a suggest the following simple strategy if you are confident that the series is geometric: Write out the first few terms

$$\sum_{n=3}^{\infty} \frac{2^{n+1}}{3^{2n}} = \frac{2^4}{3^6} + \frac{2^5}{3^8} + \cdots .$$

If we compare this to

$$\sum_{n=0}^{\infty} ar^n = a + r + \cdots ,$$

we see that the first term should be a and the second term should be r. Thus we find that

$$a = \frac{2^4}{3^6}, \qquad ar = \frac{2^5}{3^8}$$

at which point we immediately get a and can solve for $r = 2/9$. Thus

$$\sum_{n=3}^{\infty} \frac{2^{n+1}}{3^{2n}} = \frac{2^4/3^6}{1-2/9} = \frac{16}{567}.$$

The second theorem we need is

Theorem 2 (p-series test)**.** *The following series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ *and diverges if* $p \leq 1$.

Exercises

1. Compute the value of

$$\sum_{n=2} \frac{(-1)^n 2^{2n+1}}{7^n}.$$

2. Does the following series converge:

$$\sum_{n=1}^{\infty} \frac{n^e}{n^\pi}?$$

Divergence Test

Suppose that $\sum_n a_n$ converges. This means that the partial sums S_N approach some number L as N gets large. As such, we may look at the following difference

$$\begin{aligned} a_N &= (a_1 + \cdots + a_N) - (a_1 + \cdots + a_{N-1}) \\ &= S_N - S_{N-1}. \end{aligned}$$

As N gets really big, the right hand side approaches $L - L$ which is zero. Thus we see that

$$\lim_{N\to\infty} a_N = 0.$$

Morally, if we want to add up infinitely many things and not have it "blow up" then the things we are adding must be getting smaller, or approaching zero.

Turning this observation into a theorem, we get the Divergence Test.

Theorem 3 (Divergence Test)**.** *If* $\lim_{n\to\infty} a_n \neq 0$ *then the sequence* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Exercises

1. Show that the series
$$\sum_{n=1}^{\infty}(-1)^n$$
diverges.

2. Show that the series
$$\sum_{n=1}^{\infty}\frac{n^n}{n!}$$
diverges.

Integral Test

Recall the integral test: Suppose that our sequence a_n is given explicitly via the formula
$$a_n = f(n)$$
where f satisfies the following properties

(i) ______________________

(ii) ______________________

(iii) ______________________

then
$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)\mathrm{d}x \text{ converges}$$
Note: Here $\iff$ means "if and only if." Thus this theorem tells us that $\int_1^{\infty} f(x)\mathrm{d}x$ and $\sum_n a_n$ have the same behaviour in regards to convergence and divergence.

This theorem is a consequence of our earlier work on when the left and right endpoints of a Riemann sum are over and underestimates for an integral.

Exercises

1. Does the following series converge:
$$\sum_{n=10}^{\infty}\frac{1}{n\ln(n)\ln(\ln(n))^2}?$$
Hint: Consider a u-substitution $u = \ln(\ln(x))$.

2. The following integral does not converge:

$$\int_1^\infty \frac{\cos(x)}{x}\mathrm{d}x.$$

What can we therefore say about

$$\sum_{n=1}^\infty \frac{\cos(n)}{n}?$$

(i) It converges, (ii) it diverges or (iii) nothing.

3. Prove the p-series test.

Root and Ratio Tests

Recall that the root test says: Suppose we have a series $\sum_n a_n$ and

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

Then if

(i) $L >$ _____ the series diverges

(ii) $L <$ _____ the series converges absolutely

(iii) $L =$ _____ the test is inconclusive.

The ratio test has the same conclusions but instead we compute

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L.$$

Exercises

1. Determine whether the following series converges or diverges:

$$\sum_{n=1}^\infty \frac{n^n}{(2n)!}.$$

2. Determine for which p the following series converges or diverges:

$$\sum_{n=1}^\infty \left(\frac{n}{n+1}\right)^{n^p}.$$

Alternating Series Test

Recall that the alternating series test says: Suppose we have an alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

(so in particular $a_n \geq 0$). Then if

(i) a_n are eventually __________

(ii) $\lim_{n\to\infty} a_n =$ _____

then $\sum_n (-1)^n a_n$ converges.

This will almost always show up in questions of conditional convergence. In practice to check condition (i) you will need to take derivatives. You should also remember for the exam that

$$\cos(n\pi) = (-1)^n$$

Exercises

1. Determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)n}{2^n}.$$

2. Determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n^2}.$$

Note: The alternating series test can never show divergence. If you try the alternating series test and part (ii) fails, then you are likely wanting to quote the divergence test.

Comparison of Series

Simple Comparison Test

Simple comparison tells us that if $a_n \leq b_n$ for all n then

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n.$$

We can use this in the following ways.

(i) To show that $\sum_n a_n$ *diverges to* ∞ we need to find a sequence b_n such that

(a) the b_n are a lower bound, i.e. $b_n \leq a_n$

(b)
$$\sum_{n=1}^{\infty} b_n = \infty.$$

Then we have that that
$$\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} b_n = \infty,$$
so that $\sum_n a_n = \infty$ and therefore diverges.

(ii) To show that $\sum_n a_n$ *diverges to* $-\infty$ we need to find a sequence b_n such that

(a) the b_n are an upper bound, i.e. $a_n \leq b_n$

(b)
$$\sum_{n=1}^{\infty} b_n = -\infty.$$

Then we have that that
$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n = -\infty,$$
so that $\sum_n a_n = -\infty$ and therefore diverges.

(iii) To show that a_n converges we need

(a) the a_n to be non-negative, i.e. $a_n \geq 0$

(b) a sequence $b_n \geq a_n$ such that
$$\sum_{n=1}^{\infty} b_n < \infty$$
converges.

Then $\sum_n a_n$ converges.

Exercises

1. Using simple comparison, determine whether
$$\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$$
converges or diverges.

2. Using simple comparison, determine whether

$$\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$$

converges or diverges.

Limit Comparison Test

Recall that limit comparison says: Suppose that $\sum_n a_n$ and $\sum_n b_n$ are two series such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > \underline{\qquad}.$$

Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges.}$$

A common use case in this class is that we will be given a series

$$\sum_{n=1}^{\infty} \frac{n^3 - n + 1}{n^5 + n^2 - 6}$$

summing some complicated rational function. For this, one should limit compare to the series they get by throwing away all lower order terms from the numerator and denominator. In this case, we compare to

$$\sum_{n=1}^{\infty} \frac{n^3}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Exercises

1. Determine whether

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt[3]{n^7 - n + 7}}$$

converges or diverges.
Hint: You can still throw away lower order terms.

2. Determine whether

$$\sum_{n=1}^{\infty} \frac{2^n}{n^{2+1/n}}$$

converges or diverges.