Math 1B

Midterm 1 Review

Integration

Integration by Parts

Recall that integration by parts says

$$\int f(x)g'(x)\mathrm{d}x = \underline{\hspace{6cm}}$$

and follows from the product rule for differentiation.

1. Integrate the following:

   (a)
   $$\int t^{11}\ln(t)\ \mathrm{d}t$$

   (b)
   $$\int \cos^2(x)e^x\ \mathrm{d}x$$

   Hint: You may need a trig identity at some point.

Rational Integrals

Whenever you see a rational integral

$$\int \frac{f(x)}{g(x)}\ \mathrm{d}x$$

always follow the following steps:

First, before integrating, we must rewrite the integrand $f(x)/g(x)$ in a simpler form. For this we have two steps:

1. (Long Division) If the degree of the numerator is greater than *or equal to* the degree of the denominator, do long division first to write

   $$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

   where $q(x)$ is the quotient we get from long division and $r(x)$ is the remainder we get.

2. (Partial Fractions Decomposition) With the fraction we're left with (possibly after having done long division), we apply partial fractions decomposition.

With or rewritten form, we can then integrate each piece individually. For this we will have multiple cases coming from partial fractions.

1. For terms of the form $\displaystyle\int \frac{A}{(ax+b)^n}\,\mathrm{d}x$ we can immediately integrate this using logarithms or the power rule.
2. For terms of the form $\displaystyle\int \frac{Ax+B}{(ax^2+bx+c)^n}$ we must
   (a) Complete the square of the denominator
   (b) After completing the square to look like $a(x+b/(2a))^2+d$ do a u substitution with $u = x + b/(2a)$ so that the denominator looks like $(au^2+d)^n$.
   (c) Break the remaining integral into two: One of the form $\displaystyle\int \frac{Ax}{(ax^2+b)^n}\,\mathrm{d}x$ and one of the form $\displaystyle\int \frac{B}{(ax^2+b)^n}$. The first can be solved with a u-sub of the for $u = ax^2 + b$. The second one is hard.

1. Integrate the following:
   (a)
$$\int \frac{x^4}{(x^2+1)^2}\,\mathrm{d}x$$
   (b)
$$\int \frac{x-1}{x^2-x-2}\,\mathrm{d}x$$

Trigonometric Integrals

Recall the following integration rules

- For $\displaystyle\int \cos^a(x)\sin^b(x)\,\mathrm{d}x$
  (i) If a is odd, let $u = \sin(x)$.
  (ii) If b is odd, let $u = \cos(x)$.
  (iii) If both a and b are even, convert the integral into a tan and sec integral.
  (iv) *Bonus:* Rule (i) (resp. (ii)) works best if a (resp. b) is *positive*. If both a and b are odd, follow the rule with positive exponent if one exists.
- For $\displaystyle\int \tan^a(x)\sec^b(x)\,\mathrm{d}x$
  (i) If b is even, let $u = \tan(x)$.

(ii) If b is odd, convert to a sin and cos integral.

(iii) *Bonus:* Rule (i) works best when $b \geq 2$. If b is even but is ≤ 0, then try converting to a sin and cos integral anyways as it may be simpler.

- To complete the substitutions in the above two cases, you will need to use the following trig rules

$$\cos^2(\theta) + \sin^2(\theta) = 1$$
$$1 + \tan^2(\theta) = \sec^2(\theta)$$

- If none of the above apply, you may need to use one of the three double angle formulas:

$$\cos(2\theta) = 2\cos^2(\theta) - 1$$
$$\cos(2\theta) = 1 - 2\sin^2(\theta)$$
$$\sin(2\theta) = 2\sin\theta\cos\theta$$

1. Integrate the following:

(a)
$$\int \tan(x)\ \mathrm{d}x$$

(b)
$$\int \cos^3(x)\sin^{-1}(x)\ \mathrm{d}x$$

(c)
$$\int \tan^3(x)\sec^4(x)\ \mathrm{d}x$$

Trig Substitutions

Recall the following integration rules

- For $\displaystyle\int x^c(\sqrt{x^2 - r^2})^d\ \mathrm{d}x$ let $x = r\sec(x)$
- For $\displaystyle\int x^c(\sqrt{x^2 + r^2})^d\ \mathrm{d}x$ let $x = r\tan(x)$
- For $\displaystyle\int x^c(\sqrt{r^2 - x^2})^d\ \mathrm{d}x$ let $x = r\sin(x)$ or $x = r\cos(x)$ (both work).

After making these substitutions, you will use one of the trig rules

$$\cos^2(\theta) + \sin^2(\theta) = 1$$
$$1 + \tan^2(\theta) = \sec^2(\theta)$$

to reduce the integral to a trig integral as above.

1. Integrate the following:

   (a)
   $$\int \frac{\sqrt{4-x^2}}{x}\,\mathrm{d}x$$

   (b)
   $$\int \sqrt{1-x^2}\,\mathrm{d}x$$

   (c)
   $$\int x^3(x^2+1)^{-3/2}\,\mathrm{d}x$$

Putting it All Together

You will be expected to be able to combine all the above rules along with u-substitution. Here are some practice problems along these lines.

1. Integrate the following:

   (a)
   $$\int \frac{\ln(\tan x)}{\sin x\cos x}\,\mathrm{d}x$$

   (b)
   $$\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,\mathrm{d}x$$

   (c)
   $$\int \frac{1}{1+2e^x-e^{-x}}\,\mathrm{d}x$$

   (d)
   $$\int \frac{1+\sin x}{1-\sin x}\,\mathrm{d}x$$

   (e)
   $$\int x\sin^2(x)\cos(x)\,\mathrm{d}x$$

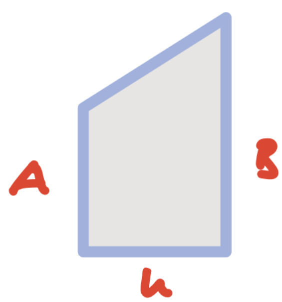
Approximate Integration

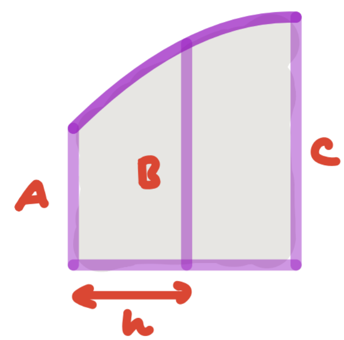
Approximation Formulas

Suppose we want to approximate $\displaystyle\int_a^b f(x)\,\mathrm{d}x$. Three simple ways are the left, right and midpoint approximations. Suppose we use a step-size of n and place $n+1$ evenly spaced points $x_0,\ldots,x_n$ in the interval $[a,b]$. Then the left, right and midpoint approximations have the following formulas:

- $L_n =$ ________________________________
- $R_n =$ ________________________________
- $M_n =$ ________________________________

More complex approximations are the trapezoid rule and Simpson's rule which use trapezoids and parabolas respectively to approximate the integral. To help derive the formulas, recall the areas of the following two shapes:





Area = ________________ Area = ________________

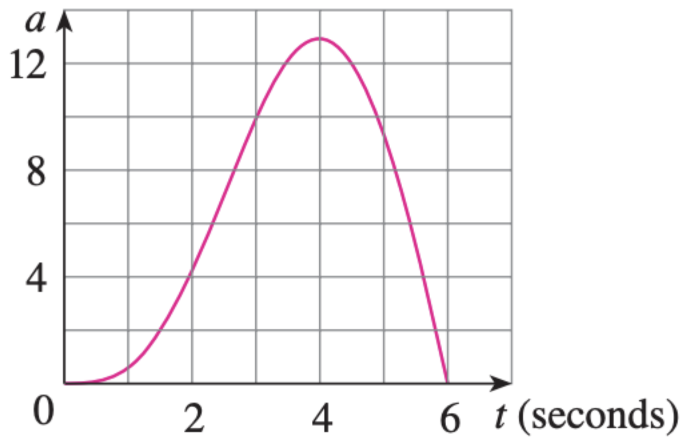
Using this, we may re-derive the following formulas for the trapezoid and Simpson's rule.

- $T_n =$ ________________________________
- $S_n =$ ________________________________

Note: n must be even for Simpson's rule. (Why?)

1. Compute L_4, R_4, M_4, T_4 and S_4 for
$$\int_0^8 \cos^2(\pi x/2)\, \mathrm{d}x.$$

2. Using the graph



estimate $\int_0^6 a(t)\, dt$ by computing L_3, T_3 and S_6.

Error Formulas

When approximating the integrals using the above rules, we often would like to know how good of an approximation we are getting. Clearly if we increase n, we are getting better and better approximations, but how fast do they get close to the real value, and how big do we need to make n to ensure we are within a specified error?

For this, given an integral $\int_a^b f(x)\, dx$, we define the following error quantities:

- $E_{L_n} =$ ____________________________
- $E_{R_n} =$ ____________________________
- and so on for each approximation rule...

These are simply *definitions*, but we then have *theorems* which help us bound the error bounds. These are as follows:

- Suppose $|f''(x)| \leq K$ on $a \leq x \leq b$, then
  1. $|E_{M_n}| \leq$ ____________________________
  2. $|E_{T_n}| \leq$ ____________________________
- Suppose $|f^{(4)}(x)| \leq K$ on $a \leq x \leq b$, then
  1. $|E_{S_n}| \leq$ ____________________________.

Notice that Simpson's rule requires bounding the fourth derivative while the trapezoid and midpoint rule only require bounding the second derivative.

To find K we will need to use the *triangle inequality*. For example, suppose we find that

$$f''(x) = e^x(x^3 - 2x) + \cos(x)$$

and our integral is over $0 \leq x \leq 1$. Then we have (steps explained after)

$$\begin{aligned} |f''(x)| &= |e^x(x^3 - 2x) + \cos(x)| \\ &\leq |e^x(x^3 - 2x)| + |\cos(x)| \\ &\leq |e^x| \cdot |x^3| + |e^x| \cdot |2x| + |\cos(x)| \\ &\leq e^1 \cdot 1^3 + e^1 \cdot 2 + 1 \\ &= 3e + 1. \end{aligned}$$

The final expression does not depend on x and is just a number, so we may take $K = 3e+1$. The steps we used line-by-line were

(i) The first line is just by definition of $f''(x)$

(ii) The second line follows from the first by the triangle inequality

(iii) The third line follows by another triangle inequality

(iv) To get the last line, we noted that if $0 \leq x \leq 1$, then $|e^x|$ is at most e^1, $|x^3|$ is at most 1, $|2x|$ is at most 2, and $|\cos(x)|$ is at most 1.

1. If $f(x) = -x^3\cos(x)+\cos(x^2)-x$ then find a K such that $|f(x)| \leq K$ for $-1 \leq x \leq 2$.
2. Bound E_{S_6} for the integral
$$\int_0^2 xe^{x^2}\,\mathrm{d}x.$$
3. Is it possible that $M_{100} = 1$ for
$$\int_0^1 x^2\,\mathrm{d}x?$$
4. Find an n such that $|E_{M_n}| \leq 0.001$ for
$$\int_0^1 0e^{-x^3}\,\mathrm{d}x$$
and prove that the error is within the desired range for this n.