Homework 7

This homework is optional.
If you turn it in, it will be graded and only counted if it improves your homework average.
If chose to submit it, it is due by Thursday August 8th at 11:59pm on Gradescope.

The following textbook problems are suggested for review but should not be submitted:

7.6, 7.7, 7.9, 7.11.

Exercise 1. Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ given by

$$f_n(x) = 1_{[n,\infty)}(x) = \begin{cases} 1 & \text{if } x \geq n \\ 0 & \text{if } x < n. \end{cases}$$

Find a function $f : \mathbb{R} \to \mathbb{R}$ such that $f_n \to f$ pointwise. Is this convergence uniform?

Exercise 2. Suppose that $f_1 \geq f_2 \geq f_3 \geq \cdots$ are continuous functions $[a, b] \to \mathbb{R}$ with $f_n \to 0$ pointwise. Does it follow that

$$\int_a^b f_n \, \mathrm{d}x \to 0?$$

Prove or give a counterexample.

Exercise 3. We say that a sequence $(f_n)_n$ of functions $X \to \mathbb{R}$ is *uniformly bounded* if there exists an M such that

$$\sup_{x \in X} |f_n(x)| \leq M$$

for all n. Show that a uniformly convergent sequence of bounded functions is uniformly bounded.

Exercise 4. Show that if $(f_n)_n$ and $(g_n)_n$ are sequences of bounded functions $X \to \mathbb{R}$ with $f_n \to f$ and $g_n \to g$ uniformly, then $f_n g_n \to fg$ uniformly. Show that this is not necessarily the case if we drop the assumption that the f_n's and g_n's are bounded.

Bonus Questions

These questions are based on the non-examinable content we encountered at the end of class. You do not need to turn them in.

Exercise 5 (†). Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function such that for every $x \in \mathbb{R}$ there exists an $n \in \mathbb{N}$ with $f^{(n)}(x) = 0$. Show that f is a polynomial.