Homework 6

Due by Thursday August 1st at 11:59pm on Gradescope.

The following textbook problems are suggested for review but should not be submitted:

5.2, 5.15, 5.25, 5.27, 6.4, 6.5, 6.8, 6.10.

Exercise 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $|f'(x)| \leq A$ for some constant $A < 1$. Then, given any $a \in \mathbb{R}$, show that the sequence $(x_n)_n$ defined by

$$x_{n+1} = f(x_n), \quad x_0 = a$$

converges to some $x \in \mathbb{R}$. Moreover, show that this x satisfies $f(x) = x$.

Solution. By a previous homework problem, we know that f is A-Lipschitz. Thus, notice that if $n \geq 1$ then we have that

$$\begin{aligned} |x_{n+1} - x_n| &= |f(x_n) - f(x_{n-1})| \\ &\leq A|x_n - x_{n-1}|. \end{aligned}$$

By induction this tells us that

$$|x_{n+1} - x_n| \leq A^n |x_1 - x_0|.$$

It follows that for $n \geq m$ we have that

$$\begin{aligned} |x_{n+1} - x_m| &\leq |x_{n+1} - x_n| + |x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m| \\ &\leq (A^n + A^{n-1} + \cdots + A^m) \cdot |x_1 - x_0| \\ &= |x_1 - x_0| \cdot \sum_{k=m}^{n} A^k. \end{aligned} \tag{†}$$

Since $A < 1$, $\sum_n A^n$ is a convergent geometric series so the right hand side of (†) may be made arbitrarily small provided n, m are sufficiently large. Thus $(x_n)_n$ is Cauchy and hence convergent.

Say that $x_n \to x$. Since $x_{n+1} = f(x_n)$, taking limits of both sides we get that

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(x)$$

where for the second inequality we used that f is continuous. □

Exercise 2. Consider the function

$$f(x) = \begin{cases} 1 & x = 0 \\ \frac{1}{q} & x = p/q,\ \ p, q \in \mathbb{Z},\ \ p/q \text{ in reduced form},\ \ q > 0 \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is integrable on any interval and compute $\displaystyle\int_a^b f \, \mathrm{d}x$.

Solution. Fix $\varepsilon > 0$. Choose N sufficiently large so that $1/N < \varepsilon$. Let

$$S_N = \left\{ \frac{p}{q} : p, q \in \mathbb{Z},\ \ |q| \leq N \right\}.$$

Then for $x \in \mathbb{R} \setminus S_N$ we have that $|f(x)| \leq 1/N < \varepsilon$.

Now, given two distinct $x, y \in S_N$ we have that $|x - y| \geq 1/N^2$. Thus as $[a, b]$ is finite length, we must have $S_N \cap [a, b]$ is finite so let

$$S_N \cap [a, b] = \{q_1, \ldots, q_\ell\}.$$

Now choose a partition $P = \{x_0, \ldots, x_n\}$ of S_N such that every q_j belongs in (x_{i-1}, x_i) for some i and for all $i = 1, \ldots, n$ if $q_j \in [x_{i-1}, x_i]$ for some j then $\Delta x_i < \varepsilon/\ell$.

Let A be the set of indices i such that $[x_{i-1}, x_i]$ intersects S_N and B the remaining indices. If $i \in A$ then we have that

$$\sup_{x \in [x_{i-1}, x_i]} f(x) \cdot \Delta x_i \leq \varepsilon/\ell$$

since $\sup f(x) \leq 1$. If $i \in B$ then we have that

$$\sup_{x \in [x_{i-1}, x_i]} f(x) \cdot \Delta x_i \leq \varepsilon \cdot \Delta x_i.$$

Thus

$$\begin{aligned} U(P, f) &= \sum_{i \in A} \sup_{x \in [x_{i-1}, x_i]} f(x) \cdot \Delta x_i + \sum_{i \in B} \sup_{x \in [x_{i-1}, x_i]} f(x) \cdot \Delta x_i \\ &\leq |A| \varepsilon/\ell + \varepsilon \sum_{i \in B} \Delta x_i \\ &\leq \varepsilon + \varepsilon(b - a). \end{aligned}$$

Finally, by the density of irrationals we know that $L(P, f) = 0$. Thus

$$0 \leq U(P, f) - L(P, f) \leq \varepsilon \cdot (1 + b - a).$$

Since $\varepsilon > 0$ was arbitrary we see that f is integrable.

Now, by the density of the irrationals we know that $L(P, f) = 0$ for every partition P. Thus

$$\int_a^b f \, dx = \underline{\int_a^b} f, dx = \sup_P L(P, f) = 0. \qquad \square$$

Exercise 3. For a subset $A \subseteq \mathbb{R}$ let

$$1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

and set

$$g_n(x) = n \cdot 1_{(0,1/n)}(x) - (n+1) \cdot 1_{(0,1/(n+1))}(x).$$

Consider the function $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) = \sum_{n=1}^{\infty} g_n(x).$$

(i) Show that f is well-defined and $f(x) = 1_{(0,1)}(x)$.

(ii) Show that 1_I for I an interval is integrable.

(iii) Compute

$$\int_0^1 f \, dx$$

and

$$\sum_{n=1}^{\infty} \int_0^1 g_n \, dx$$

and show they are not equal. Deduce that integration does not always commute with infinite sums.

Solution. (i) f is defined via an infinite series, so we need to check that this converges for f to be well-defined. To do this, we look at the partial sums. Notice that

$$\begin{aligned}\sum_{k=0}^{n} g_k(x) &= (1_{(0,1)}(x) - 2 \cdot 1_{(0,1/2)}(x)) + (2 \cdot 1_{(0,1/2)}(x) - 3 \cdot 1_{(0,1/3)}(x)) + \\ &\quad \cdots + (n \cdot 1_{(0,1/n)}(x) - (n+1) \cdot 1_{(0,1/(n+1))}(x)) \\ &= 1_{(0,1)}(x) - (n+1) \cdot 1_{(0,1/(n+1))}(x),\end{aligned} \tag{‡}$$

since the sum is telescoping. Now, for a fixed $x \in [0, 1]$, when n is sufficiently large we have that $x \notin (0, 1/n)$ so

$$n \cdot 1_{(0,1/n)}(x) = 0.$$

Thus $\sum_n g_n(x)$ converges and $f(x) = 1_{(0,1)}(x)$ by (‡).

(ii) If I is an interval, then 1_I has at most two discontinuities occuring at the potential endpoints of I. Thus 1_I is integrable.

(iii) Since 1 and $1_{(0,1)}(x)$ disagree at only two values on $[0, 1]$ we have that

$$\int_0^1 f \, dx = \int_0^1 1 \, dx = 1.$$

On the other hand, notice that

$$\begin{aligned}\int_0^1 1_{(0,1/n)}(x) \, dx &= \int_0^{1/n} 1 \, dx + \int_{1/n}^1 0 \, dx \\ &= \frac{1}{n}.\end{aligned}$$

Thus

$$\int_0^1 g_n \, dx = n \cdot \frac{1}{n} - (n+1) \cdot \frac{1}{n+1} = 0.$$

It follows that

$$\int_0^1 \sum_{n=1}^{\infty} g_n(x) \, dx = 1 \neq 0 = \sum_{n=1}^{\infty} \int_0^1 g_n(x) \, dx$$

so integrals do not necessarily commute with infinite sums. □

Exercise 4. Show that there is no differentiable function $f : \mathbb{R} \to \mathbb{R}$ with integrable derivative which satisfies $f(x)f'(x) = 1$ for all x.

Solution. Suppose f were such a function. For $x < 0$ we get by integration by parts that

$$\int_x^0 f(t)f'(t)\,\mathrm{d}t = f(0)^2 - f(x)^2 - \int_x^0 f'(t)f(t)\,\mathrm{d}t.$$

Thus

$$f(0)^2 - f(x)^2 = 2\int_x^0 f(t)f'(t)\,\mathrm{d}t = -2x$$

where the last equality is because $f(t)f'(t) = 1$. Hence

$$f(x)^2 = 2x + f(0)^2$$

but letting x be sufficiently negative, the left hand side in non-negative whereas the right hand side is negative, a contradiction. □

Exercise 5. Show that if $f : [a, b] \to \mathbb{R}$ is continuous with $f \geq 0$ and $\displaystyle\int_a^b f\,\mathrm{d}x = 0$ then $f = 0$.

Solution. Suppose that $x_0 \in [a, b]$ is such that $f(x_0) > 0$. Then as f is continuous, we may find an interval $[c, d] \subseteq [a, b]$ of positive length containing x_0 on which $f > f(x_0)/2$. Then

$$f(x) \geq \frac{f(x_0)}{2} \cdot 1_{[c,d]}(x)$$

on $[a, b]$ so

$$\int_a^b f,\mathrm{d}x \geq \frac{f(x_0)}{2}\int_a^b 1_{[c,d]}(x)\,\mathrm{d}x = \frac{f(x_0)}{2} \cdot (d - c) > 0$$

a contradiction. □

Exercise 6. Suppose $f : [a, b] \to \mathbb{R}$ is integrable and $f(x) = 0$ for all $x \in [a, b] \cap \mathbb{Q}$. Show that $\displaystyle\int_a^b f\,\mathrm{d}x = 0$.

Solution. Fix $\varepsilon > 0$. Then there exists a partition $P = \{x_0, \ldots, x_n\}$ with $U(P, f) - L(P, f) < \varepsilon$. Since the rationals are dense, for each $i = 1, \ldots, n$ we may find a rational $t_i \in [x_{i-1}, x_i]$. We then have that

$$\sum_{i=1}^n f(t_i)\Delta x_i = 0$$

since $f(t_i) = 0$ for all i. But

$$\left|\sum_{i=1}^n f(t_i)\Delta x_i - \int_a^b f\,\mathrm{d}x\right| < \varepsilon$$

so

$$\left| \int_a^b f \, dx \right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the result follows. □