Homework 6

Due by Thursday August 1st at 11:59pm on Gradescope.

The following textbook problems are suggested for review but should not be submitted:

5.2, 5.15, 5.25, 5.27, 6.4, 6.5, 6.8, 6.10.

Exercise 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $|f'(x)| \leq A$ for some constant $A < 1$. Then, given any $a \in \mathbb{R}$, show that the sequence $(x_n)_n$ defined by

$$x_{n+1} = f(x_n), \quad x_0 = a$$

converges to some $x \in \mathbb{R}$. Moreover, show that this x satisfies $f(x) = x$.

Exercise 2. Consider the function

$$f(x) = \begin{cases} 1 & x = 0 \\ \frac{1}{q} & x = p/q, \; p, q \in \mathbb{Z}, \; p/q \text{ in reduced form}, \; q > 0 \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is integrable on any interval and compute $\displaystyle\int_a^b f \, \mathrm{d}x$.

Exercise 3. For a subset $A \subseteq \mathbb{R}$ let

$$1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

and set

$$g_n(x) = n \cdot 1_{(0,1/n)}(x) - (n+1) \cdot 1_{(0,1/(n+1))}(x).$$

Consider the function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \sum_{n=1}^{\infty} g_n(x).$$

(i) Show that f is well-defined and $f(x) = 1_{(0,1)}(x)$.

(ii) Show that 1_I for I an interval is integrable.

(iii) Compute

$$\int_0^1 f \, \mathrm{d}x$$

and

$$\sum_{n=1}^{\infty} \int_0^1 g_n \, \mathrm{d}x$$

and show they are not equal. Deduce that integration does not always commute with infinite sums.

Exercise 4. Show that there is no differentiable function $f : \mathbb{R} \to \mathbb{R}$ with integrable derivative which satisfies $f(x)f'(x) = 1$ for all x.

Exercise 5. Show that if $f : [a, b] \to \mathbb{R}$ is continuous with $f \geq 0$ and $\displaystyle\int_a^b f \, \mathrm{d}x = 0$ then $f = 0$.

Exercise 6. Suppose $f : [a, b] \to \mathbb{R}$ is integrable and $f(x) = 0$ for all $x \in [a, b] \cap \mathbb{Q}$. Show that $\displaystyle\int_a^b f \, \mathrm{d}x = 0$.