Homework 4

Due by Thursday July 18th at 11:59pm on Gradescope.

The following textbook problems are suggested for review but should not be submitted:

3.6, 3.7, 3.9, 3.22 (hard), 3.23, 4.1, 4.2.

Exercise 1. Suppose that we have a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots . \tag{†}$$

We will call a *regrouping* of $\sum_n a_n$ to be any series arising by grouping terms in the summation, e.g.

$$a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + a_8 + a_9 + (a_{10} + a_{11}) + \cdots$$

and

$$(a_1 + a_2 + a_3) + a_4 + a_5 + a_6 + \cdots$$

are examples of regroupings of (†).

(i) Show that if $\sum_n a_n$ converges, then every regrouping converges to the same value.

(ii) Give an example of a divergent series $\sum_n a_n$ which has a convergent regrouping.

Exercise 2. Suppose that $\sum_n (-1)^n 2^n a_n$ converges, does it follow that $\sum_n a_n$ converges? Prove or give a counterexample.

Exercise 3. Compute the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{n!}{n^n} \cdot z^n.$$

Hint: You may use that

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Exercise 4. Let $f : X \to \mathbb{R}$ be a continuous function. Define

$$Z(f) = \{x \in X : f(x) = 0\}$$

to be the *zero set* of f. Show that $Z(f)$ is closed.

Exercise 5. Let X, Y be metric spaces. Show that the two projection maps

$$\begin{aligned} \pi_X : X \times Y &\longrightarrow X \\ (x, y) &\longmapsto x \end{aligned}$$

and

$$\begin{aligned} \pi_Y : X \times Y &\longrightarrow Y \\ (x, y) &\longmapsto y \end{aligned}$$

are continuous.