Homework 3

Due by Thursday July 11th at 11:59pm on Gradescope.

The following textbook problems are suggested for review but should not be submitted:

3.2, 3.3, 3.4.

Exercise 1. (i) For every n, let S_n be the set of all rational numbers that can be represented as a fraction p/q with $|q| \leq n$. Show that S_n is complete.

(ii) Show that

$$\mathbb{Q} = \bigcup_{n \in \mathbb{N}} S_n.$$

(iii) Show that $\mathbb{Q}$ is not complete.

Exercise 2. Let $(s_n)_n$ be a sequence of real numbers. Show that if $(s_n)_n$ converges then $(|s_n|)_n$ converges. Given an example to show that the converse is not true.

Exercise 3. Give an example of two sequences $(s_n)_n$ and $(t_n)_n$ of real numbers such that

$$\liminf_n (s_n t_n) \neq \liminf_n s_n \cdot \liminf_n t_n$$

where all terms in the above are finite (i.e. not $\pm\infty$).

Exercise 4. Show that for two real sequences $(s_n)_n$ and $(t_n)_n$ that

$$\limsup_n (s_n + t_n) \leq \limsup_n s_n + \limsup_n t_n$$

(whenever the right hand side is not of the form $\infty - \infty$).

Exercise 5. Consider the sequence defined by

$$s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

(i) Show that $(s_n)_n$ is monotone.

(ii) Show that

$$s_{2^{n+1}} - s_{2^n} \geq \frac{1}{2}.$$

Deduce that $(s_n)_n$ is unbounded.

(iii) Deduce from (i) and (ii) that $(s_n)_n$ does not converge.