Midterm 2

The exam will last 90 minutes. Do not begin until instructed.

Please write your answers legibly in the space provided under each question, crossing out any work you do not want graded. Extra paper and/or scratch paper may be provided upon request.

Name: ______________________________

SID: ______________________________

Problem 1. Determine whether the following series are convergent or divergent:

(i) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$

(ii) $\displaystyle\sum_{n=0}^{\infty}\frac{3^n}{4^n+5^n}$

(iii) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\cdot\frac{n}{n+1}$

[You may use that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.]

Solution. (i) We have that

$$\limsup_n \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_n \left(\frac{n}{n+1}\right)^n = \lim_n \frac{1}{(1+1/n)^n} = 1/e$$

and $1/e < 1$ so the series converges by the root test.

(ii) We have that

$$\frac{3^n}{4^n+5^n} \leq \frac{3^n}{4^n}$$

and $\sum_n (3/4)^n$ converges by the geometric series test, so the original series converges by comparison.

(iii) We have that

$$\lim_n \left|(-1)^n\cdot\frac{n}{n+1}\right| = 1 \neq 0$$

so the series diverges by the divergence test. □

Problem 2. Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot \cdots \cdot 2n}{n^n} \cdot z^n$$

[You may use that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.]

Solution. We have that

$$\begin{aligned}
\lim_n \frac{\frac{2\cdot 4\cdot\cdots\cdot(2n)\cdot(2(n+1))}{(n+1)^{n+1}}}{\frac{2\cdot 4\cdot\cdots\cdot(2n)}{n^n}} &= \lim_n \frac{2(n+1)}{n+1} \cdot \left(\frac{n}{n+1}\right)^n \\
&= 2 \lim_n \frac{1}{(1+1/n)^n} \\
&= 2/e.
\end{aligned}$$

Thus by the ratio test, our power series converges when $|z| < e/2$ and diverges when $|z| > e/2$. Since this is the defining property of the radius of convergence, we have that $R = e/2$. □

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} 1 & x = 0 \\ \frac{1}{q} & x = p/q \text{ where } p/q \text{ is in reduced form, } q > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Show that for all $a \in \mathbb{R}$, $\lim_{x \to a} f(x) = 0$.

(ii) Conclude that f is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous on $\mathbb{Q}$.

Solution. (i) Fix $\varepsilon > 0$. Given any two fractions $a/b \neq p/q$ with $|b|, |q| \leq N$ we have that

$$\begin{aligned} \left| \frac{a}{b} - \frac{p}{q} \right| &= \frac{|aq - bp|}{bq} \\ &\geq \frac{1}{N^2}. \end{aligned}$$

That is to say any two distinct elements of the set

$$S_N = \left\{ \frac{p}{q} : p, q \in \mathbb{Z},\ |q| \leq N \right\}$$

are distance at least $1/N^2$ apart.

In particular, take $N \in \mathbb{N}$ to be such that $1/N < \varepsilon$. By the above, any finite length interval centered at a will contain at most finitely many elements of S_N, so we may shrink it to find a $\delta > 0$ such that

$$(a - \delta, a + \delta) \cap S_N \subseteq \{a\},$$

i.e. the only element of $(a - \delta, a + \delta)$ in S_N is potentially a itself. But then

$$\begin{aligned} 0 < |x - a| < \delta &\implies x \notin S_N \\ &\implies |f(x)| < 1/N < \varepsilon \end{aligned}$$

as required, so $\lim_{x \to a} f(x) = 0$.

(ii) f is continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$. By part (i), this is if and only if $f(a) = 0$. Thus f is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous on $\mathbb{Q}$. □

Problem 4. Let $f : X \to Y$ be a function and $f(X) \subseteq Z \subseteq Y$. Show that f is continuous when viewed as a function $X \to Z$ if and only if f is continuous when viewed as a function $X \to Y$.

Solution 1. Write f_Z for f viewed as a function $X \to Z$ and f_Y for f viewed as a function $X \to Y$. The open subsets of Z are of the form $Z \cap V$ for V open in Y, and

$$f_Z^{-1}(Z \cap V) = f_Y^{-1}(V)$$

since $Z \supseteq f(X)$.

Thus the pullback under f_Z of every open in Z is open if and only if the pullback under f_Y of every open in Y is open, which is to say f_Z is continuous if and only if f_Y is continuous. □

Solution 2. Let $(x_n)_n$ be a sequence converging to x in X. Since $Z, Y \supseteq f(X)$, checking whether $f(x_n) \to f(x)$ may be done inside either Z or Y and gives the same answer. Thus the sequence definition of continuity gives the result. □

Problem 5. Show that a uniformly continuous function $f : (a, b) \to \mathbb{R}$ may be extended to a continuous function $[a, b] \to \mathbb{R}$, i.e. there exists a continuous function $g : [a, b] \to \mathbb{R}$ with $g|_{(a,b)} = f$.

[Hint: You may use the fact that if $f : X \to Y$ is uniformly continuous and $(x_n)_n$ is a Cauchy sequence in X, then $(f(x_n))_n$ is Cauchy.]

Solution. Let $(a_n)_n$ be any sequence in (a, b) which converges to a in $\mathbb{R}$, e.g. $(a + (b - a)/n)_n$. Then $(a_n)_n$ is Cauchy, so $(f(a_n))_n$ is Cauchy by the hint as f is assumed to be uniformly continuous. Thus $(f(a_n))_n$ converges in $\mathbb{R}$ as $\mathbb{R}$ is complete. Similarly, let $(b_n)_n$ be any sequence in (a, b) converging to b in $\mathbb{R}$.

Now define

$$g(x) = \begin{cases} f(x) & x \in (a, b) \\ \lim_n f(a_n) & x = a \\ \lim_n f(b_n) & x = b. \end{cases}$$

Then $g : [a, b] \to \mathbb{R}$ restricts to f on (a, b) by definition, so we need only show that g is continuous. g is continuous at every $x \in (a, b)$ since f is, so we need only check that g is continuous at a and b. We will check continuity at a since the proof for b is similar.

Let $(x_n)_n$ be a sequence in $[a, b]$ with $x_n \to a$. We may WLOG assume that $x_n \in (a, b)$ for all n. Then the sequence

$$x_1, a_1, x_2, a_2, x_3, a_3, \ldots$$

converges to a in $\mathbb{R}$, hence is Cauchy. Thus

$$f(x_1), f(a_1), f(x_2), f(a_2), \ldots$$

is Cauchy by the hint, hence convergent in $\mathbb{R}$. But we already know that the subsequence $f(a_n) \to g(a)$ by definition, so the subsequence $(f(x_n))_n$ must also converge to $g(a)$. But then

$$g(x_n) = f(x_n) \to g(a)$$

as required. Thus g is continuous at a. □

This page is scratch paper. Do not write answers you want graded here unless you explicitly denote on the designated problem page that there is extra work here. In this case, you should clearly separate it from scratch work.