Midterm 2 Practice Questions

Note: This is to help study for Midterm 2. While the questions are similar to what a real exam may contain they may lean on the harder side. This should be treated as a study guide and not a mock exam. More care will be taken to make sure the real exam is manageable both in difficulty and amount of time required to complete.

Problem 1. Determine whether the following series are convergent or divergent:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^3 + 1}$

(ii) $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(2n)}{1 + 2^n}$

(iii) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \sin(n)$

Solution. (i) When n is large, we have that $(n^2 - 1)/(n^3 + 1) \sim n^2/n^3 = 1/n$ so we expect that this series to diverge since $\sum_n 1/n$ diverges. For this, we have that

$$\begin{aligned} \frac{n^2 - 1}{n^3 + 1} &\geq \frac{1}{2} \cdot \frac{n^2 - 1}{n^3} \\ &= \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n^3}\right). \end{aligned}$$

By the p-series test, $\sum_n (1/n - 1/n^3)$ is a sum of a divergent series and a convergent series, hence divergent. Thus the original series must diverge by comparison.

(ii) We have that

$$\begin{aligned} \left|\frac{\sin(2n)}{1 + 2^n}\right| &\leq \frac{1}{1 + 2^n} \\ &\leq \frac{1}{2^n} \end{aligned}$$

and $\sum_n 2^{-n}$ converges by the geometric series test. Hence our original series converges absolutely by comparison.

(iii) We have that $\lim_{n\to\infty} (-1)^n \sin(n)$ does not exist, so our series diverges by the divergence test. □

Problem 2. Find the radius of convergence of the following power series:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^n} \cdot z^n$

(ii) $\displaystyle\sum_{n=0}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2} z^n$

[You may use that $\lim_{n\to\infty}(1 + \frac{1}{n})^n = e$.]

Solution. (i) We have that

$$\begin{aligned}\limsup_n \left(\frac{2^n}{n^n}\right)^{1/n} &= \limsup_n \frac{2}{n} \\ &= \lim_n \frac{2}{n} \\ &= 0\end{aligned}$$

so $R = \infty$.

(ii) We have that

$$\begin{aligned}\limsup_n \left(\left(1 + \frac{1}{n}\right)^{n^2}\right)^{1/n} &= \limsup_n \left(1 + \frac{1}{n}\right)^n \\ &= \lim_n \left(1 + \frac{1}{n}\right)^n \\ &= e\end{aligned}$$

so $R = 1/e$. □

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{x} & x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & x \in \mathbb{Q}. \end{cases}$$

Determine where f is continuous.

Solution. Suppose that f is continuous at $x = a$. Then since $\mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{Q}$ are dense in $\mathbb{R}$, we may find a sequence $(r_n)_n$ of irrationals with $r_n \to a$ and a sequence of rationals $(q_n)_n$ with $q_n \to a$. Then since f is continuous at a, we must have that $f(q_n) \to f(a)$ and $f(r_n) \to f(a)$. But

$$f(q_n) = 1 \to 1$$

and

$$f(r_n) = \frac{1}{r_n} \to \frac{1}{a}.$$

Thus

$$f(a) = 1 = \frac{1}{a}$$

which is only possible if $a = 1$.

On the other hand, we claim f is continuous at $x = 1$. To do this, fix $\varepsilon > 0$. Since $1/x$ is continuous at 1, we may find $\delta > 0$ such that

$$|x - 1| < \delta \Longrightarrow \left|\frac{1}{x} - 1\right| < \varepsilon. \tag{$\dagger$}$$

But then

$$|x - 1| < \delta \Longrightarrow |f(x) - 1| < \varepsilon \tag{$\ddagger$}$$

since either x is irrational, in which case ($\ddagger$) follows from ($\dagger$), or x is rational in which case $f(x) = 1$ so ($\ddagger$) is trivial. $\square$

Problem 4. Given any metric space X, it is a fact that the diagonal

$$\Delta = \{(x, x) : x \in X\} \subseteq X \times X$$

is closed. Using this fact, prove the following:

Let $f, g : X \to Y$ be continuous functions. Show that

$$\{x \in X : f(x) = g(x)\} \subseteq X$$

is closed.

Solution. Consider the function $h = (f, g) : X \to X \times X$ which is continuous as each component is continuous. Then we have that

$$\begin{aligned} x \in h^{-1}(\Delta) &\Longleftrightarrow h(x) = (f(x), g(x)) \in \Delta \\ &\Longleftrightarrow f(x) = g(x). \end{aligned}$$

Thus

$$\{x \in X : f(x) = g(x)\} = h^{-1}(\Delta)$$

which is the pre-image of a closed set under a continuous map, hence closed. $\square$

Problem 5. Let $f : X \to Y$ be continuous and $E \subseteq X$. Show that $f(\overline{E}) \subseteq \overline{f(E)}$ and give an example to show that this containment may be proper.

Solution. Set $S = \overline{f(E)}$ so that S is closed and $S \supseteq f(E)$. Then $f^{-1}(S)$ is closed since f is continuous, and moreover

$$f^{-1}(S) \supseteq f^{-1}(f(E)) \supseteq E.$$

Thus $f^{-1}(S)$ is a closed set containing E, so $f^{-1}(S) \supseteq \overline{E}$ by definition of $\overline{E}$. Hence

$$f(\overline{E}) \subseteq S = \overline{f(E)}$$

as required.

To see that this containment may be proper, consider the map

$$\begin{aligned} f : (\mathbb{R}, d_{\text{disc}}) &\longrightarrow \mathbb{R} \\ x &\longmapsto x \end{aligned}$$

given by the set-theoretic identity from $\mathbb{R}$ with the discrete metric to $\mathbb{R}$ with the standard metric. Then f is continuous and

$$f(\overline{(0,1)}) = f((0,1)) = (0,1)$$

but

$$\overline{f((0,1))} = \overline{(0,1)} = [0,1]. \qquad \square$$

Problem 6. Let (X, d) be a metric space and let $E \subseteq X$.

(i) Show that $\rho_E : X \to \mathbb{R}$ defined by

$$\rho_E(x) = \inf_{y \in E} d(x, y)$$

is continuous.

(ii) Show that $\{x \in X : \rho_E(x) = 0\} = \overline{E}$.

Solution. (i) Let $x, y \in X$. We have that for any $z \in E$

$$d(x, z) \leq d(x, y) + d(y, z).$$

Taking the infimum over $z \in E$ we see that

$$\begin{aligned} \rho_E(x) &= \inf_{z \in E} d(x, z) \\ &\leq \inf_{z \in E} \big(d(x, y) + d(y, z)\big) \\ &= d(x, y) + \rho_E(y). \end{aligned}$$

Hence

$$\rho_E(x) - \rho_E(y) \leq d(x, y)$$

and swapping the roles of x and y in the above argument we find that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y).$$

Thus ρ_E is 1-Lipschitz and hence continuous.

(ii) Let $Z(\rho_E) = \{x \in X : \rho_E(x) = 0\}$. Then $Z(\rho_E) \supseteq E$ by definition of ρ_E and $Z(\rho_E) = \rho_E^{-1}(\{0\})$ is closed since ρ_E is continuous. Thus $Z(\rho_E) \supseteq \overline{E}$.

On the other hand, suppose that $x \in Z(\rho_E)$. Then by definition of ρ_E we may find a sequence $(y_n)_n$ in E with $y_n \to x$. Thus $x \in \lim E = \overline{E}$ so we conclude that $Z(\rho_E) \subseteq \overline{E}$ giving the reverse containment. $\square$