Midterm 2 Practice Questions

Note: This is to help study for Midterm 2. While the questions are similar to what a real exam may contain they may lean on the harder side. This should be treated as a study guide and not a mock exam. More care will be taken to make sure the real exam is manageable both in difficulty and amount of time required to complete.

Problem 1. Determine whether the following series are convergent or divergent:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+1}$

(ii) $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(2n)}{1+2^n}$

(iii) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \sin(n)$

Problem 2. Find the radius of convergence of the following power series:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^n} \cdot z^n$

(ii) $\displaystyle\sum_{n=0}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} z^n$

[You may use that $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.]

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{x} & x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & x \in \mathbb{Q}. \end{cases}$$

Determine where f is continuous.

Problem 4. Given any metric space X, it is a fact that the diagonal

$$\Delta = \{(x,x) : x \in X\} \subseteq X \times X$$

is closed. Using this fact, prove the following:

Let $f, g : X \to Y$ be continuous functions. Show that

$$\{x \in X : f(x) = g(x)\} \subseteq X$$

is closed.

Problem 5. Let $f : X \to Y$ be continuous and $E \subseteq X$. Show that $f(\overline{E}) \subseteq \overline{f(E)}$ and give an example to show that this containment may be proper.

Problem 6. Let (X, d) be a metric space and let $E \subseteq X$.

(i) Show that $\rho_E : X \to \mathbb{R}$ defined by

$$\rho_E(x) = \inf_{y \in E} d(x, y)$$

is continuous.

(ii) Show that $\{x \in X : \rho_E(x) = 0\} = \overline{E}$.