Midterm 2

The exam will last 90 minutes. Do not begin until instructed.

Please write your answers legibly in the space provided under each question, crossing out any work you do not want graded. Extra paper and/or scratch paper may be provided upon request.

Name: ______________________________

SID: ______________________________

Problem 1. Determine whether the following series are convergent or divergent:

(i) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

(ii) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{4^n + 5^n}$

(iii) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{n+1}$

[You may use that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.]

Problem 2. Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot \cdots \cdot 2n}{n^n} \cdot z^n$$

[You may use that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.]

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} 1 & x = 0 \\ \frac{1}{q} & x = p/q \text{ where } p/q \text{ is in reduced form, } q > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Show that for all $a \in \mathbb{R}$, $\lim_{x \to a} f(x) = 0$.

(ii) Conclude that f is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous on $\mathbb{Q}$.

Problem 4. Let $f : X \to Y$ be a function and $f(X) \subseteq Z \subseteq Y$. Show that f is continuous when viewed as a function $X \to Z$ if and only if f is continuous when viewed as a function $X \to Y$.

Problem 5. Show that a uniformly continuous function $f : (a, b) \to \mathbb{R}$ may be extended to a continuous function $[a, b] \to \mathbb{R}$, i.e. there exists a continuous function $g : [a, b] \to \mathbb{R}$ with $g|_{(a,b)} = f$.

[Hint: You may use the fact that if $f : X \to Y$ is uniformly continuous and $(x_n)_n$ is a Cauchy sequence in X, then $(f(x_n))_n$ is Cauchy.]

This page is scratch paper. Do not write answers you want graded here unless you explicitly denote on the designated problem page that there is extra work here. In this case, you should clearly separate it from scratch work.