Midterm 1 Practice Questions

Note: This is to help study for Midterm 1. While the questions are similar to what a real exam may contain they may lean on the harder side. This should be treated as a study guide and not a mock exam. More care will be taken to make sure the real exam is manageable both in difficulty and amount of time required to complete.

Problem 1. For the following sets determine whether they are (i) open, (ii) closed, and/or (iii) compact. Note that multiple or none of the properties may hold in some cases. The sets will be listed in the form $Y \subseteq X$ and you should answer for the set Y viewed as a subset of X.

(i) $\mathbb{Q} \subseteq \mathbb{R}$

(ii) $[0,1] \subseteq [0,1] \cup [2,3]$

(iii) $(0,1) \subseteq \mathbb{R}$

(iv) $\mathbb{N} = \{1,2,3,4,\dots\} \subseteq \mathbb{R}$

Solution. (i) None of the three.

To see that $\mathbb{Q}$ is not open, it suffices to show that the complement is not closed. For this, notice that $(\sqrt{2}/n)_n$ is a sequence of irrational numbers converging to $0 \in \mathbb{Q}$.

To see that $\mathbb{Q}$ is not closed, it suffices to show the complement is not open. For this, recall that every open ball in $\mathbb{R}$ contains a rational number, so for example there is no open ball centered at $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ contained in $\mathbb{R} \setminus \mathbb{Q}$.

Since $\mathbb{Q}$ is not closed in $\mathbb{R}$ it cannot be compact.

(ii) All three.

To see that $[0,1]$ is closed, notice that $[0,1] = D_{1/2}(1/2)$ is a closed ball inside our space hence closed.

To see that $[0,1]$ is open, notice that $([0,1] \cup [2,3]) \setminus [0,1] = [2,3]$ which is closed by the same argument as for $[0,1]$. Hence the complement of $[0,1]$ is closed, so $[0,1]$ is open.

Being compact does not depend on the ambient space and we know all finite, closed intervals in $\mathbb{R}$ are compact so $[0,1]$ is compact.

(iii) Open only.

$(0,1) = B_{1/2}(1/2)$ is an open ball, so it is open.

Since $(0, 1)$ is open, it cannot be closed since there are no non-tirvial clopen subsets of $\mathbb{R}$. Alternatively, $(1/n)_n$ is a sequence in $(0, 1)$ which converges to $0 \notin (0, 1)$.

Since $(0, 1)$ is not closed, it cannot be compact.

(iv) Closed only.

To see that $\mathbb{N}$ is not open, notice that every positive radius ball centered at 1 contains non-integers. Alternatively, we will see that $\mathbb{N}$ is closed so it cannot be open as there are no non-trivial clopens in $\mathbb{R}$.

We have that

$$\mathbb{R} \setminus \mathbb{N} = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup \cdots$$

is open so $\mathbb{N}$ is closed.

$\mathbb{N}$ cannot be compact as it is not bounded. □

Problem 2. Let X be a metric space.

(i) Show that the singleton set $\{x\}$ is closed for all $x \in X$.

(ii) Does this imply that all finite subsets of X are closed?

Solution. (i) The only sequences in $\{x\}$ is the constant sequence which converges to x. Thus $\lim\{x\} = \{x\}$ so $\{x\}$ is closed. Alternatively, one may check from definition that $\{x\}$ is compact hence closed.

(ii) Yes because finite unions of closed sets are closed. Indeed, by writing

$$\{x_1, \dots, x_n\} = \{x_1\} \cup \cdots \cup \{x_n\}$$

and applying (i) we see that any finite set $\{x_1, \dots, x_n\}$ is closed. □

Problem 3. Show that if K and V are compact then $K \cup V$ is compact.

Solution. Let $\mathcal{C} = \{U_\alpha : \alpha \in A\}$ be an open cover of $K \cup V$. Then, in particular, $\mathcal{C}$ is also an open cover of K so we may find a finite subcover and write

$$U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \supseteq K.$$

Similarly, $\mathcal{C}$ is an open cover of V so we may write

$$U_{\alpha_{n+1}} \cup \cdots \cup U_{\alpha_m} \supseteq V.$$

Hence

$$U_{\alpha_1} \cup \cdots \cup U_{\alpha_m} \supseteq K \cup V$$

and we have found a finite subcover of $K \cup V$. $\square$

Problem 4. Given two sequences $(t_n)_n$ and $(s_n)_n$ of real numbers, is it true that

$$\limsup_n (s_n t_n) = \limsup_n s_n \cdot \limsup_n t_n?$$

Solution. No. Consider

$$s_n = (-1)^n, \quad t_n = (-1)^{n+1}.$$

Then $s_n t_n = -1$ is the constant sequence so $\limsup_n (s_n t_n) = -1$. However,

$$\limsup_n s_n = \limsup_n t_n = 1. \qquad \square$$

Problem 5. Let

$$s_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n}.$$

(i) Show that $s_1 > s_3 > s_5 > s_7 > \cdots > 0$. In other words, show that the odd terms are all positive and form a decreasing subsequence. Conclude that $\lim_n s_{2n+1}$ exists.

(ii) Show that $\lim_n (s_n - s_{n+1}) = 0$.

(iii) Use (i) and (ii) to argue that $(s_n)_n$ converges.

Solution. (i) We have that

$$\begin{aligned} s_{2n+3} &= s_{2n+1} - \frac{1}{2n+2} + \frac{1}{2n+3} \\ &= s_{2n+1} - \frac{1}{(2n+2)(2n+3)} \\ &< s_{2n+1} \end{aligned}$$

so we get that the odd terms form a decreasing sequence. However, we also have that

$$s_{2n+1} = \underbrace{\left(1 - \frac{1}{2}\right)}_{>0} + \underbrace{\left(\frac{1}{3} - \frac{1}{4}\right)}_{>0} + \cdots + \underbrace{\left(\frac{1}{2n-1} - \frac{1}{2n}\right)}_{>0} + \underbrace{\frac{1}{2n+1}}_{>0} > 0$$

so all the odd terms are positive as well. Since $(s_{2n+1})_n$ is a bounded, monotone sequence it converges.

(ii) We have that

$$s_n - s_{n+1} = \frac{(-1)^{n+1}}{n+1} \to 0.$$

(iii) Fix $\varepsilon > 0$ and suppose that $s_{2n+1} \to s$ by part (i). Let N_1 be such that for all $n \geq N_1$ we have that

$$|s_{2n+1} - s| < \varepsilon/2$$

and let N_2 be such that for all $n \geq N_2$ we have

$$|s_n - s_{n+1}| < \varepsilon/2.$$

Then for all $n \geq \max\{2N_1 + 1, N_2\}$ we have that either $n = 2k$ is even and thus

$$\begin{aligned} |s_n - s| &\leq |s_n - s_{n+1}| + |s_{n+1} - s| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

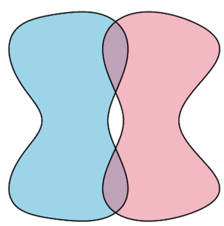
or $n = 2k + 1$ is odd and so

$$|s_n - s| < \varepsilon/2 < \varepsilon$$

as required. □

Problem 6. Is the intersection of two connected sets connected? Prove or give a counterexample.

Solution. No. Consider the following picture taking place inside $\mathbb{R}^2$:



□

Problem 7. Let $(x_n)_n$ be a sequence in a metric space X. Define the following set

$$E = \{x \in X : \exists \text{a subsequence } (x_{n_k})_k \text{ of } (x_n)_n \text{ with } x_{n_k} \to x\}.$$

Show that E is closed. [Hint: Given a sequence $(y_n)_n$ in E with $y_n \to y$, build a subsequence $(x_{n_k})_k$ of $(x_n)_n$ with the property that $d(x_{n_k}, y_k) < 1/k$ for all k and show that $x_{n_k} \to y$.]

Solution. As per the hint, let $(y_n)_n$ be a sequence in E with $y_n \to y$. We inductively build a subsequence $(x_{n_k})_k$ of $(x_n)_n$ with the property that $d(x_{n_k}, y_k) < 1/k$ so that $x_{n_k} \to y$ and hence $y \in E$. (See the proof from lectures that $\lim \lim S = \lim S$ to see why $x_{n_k} \to y$.)

To do this, notice that since $y_1 \in E$ there is a subsequence converging to y_1. In particular, we may find $n_1 \in \mathbb{N}$ such that $d(x_{n_1}, y_1) < 1$. Now suppose that $x_{n_1}, \dots, x_{n_k}$ have been chosen. Then there exists a subsequence of $(x_n)_n$ converging to y_{k+1}. Going far enough down this subsequence, we may find $n_{k+1} > n_k$ such that $d(x_{n_{k+1}}, y_{k+1}) < 1/(k+1)$ as required. This gives us our desired subsequence $(x_{n_k})_k$ converging to y. □