Midterm 1 Practice Questions

Note: This is to help study for Midterm 1. While the questions are similar to what a real exam may contain they may lean on the harder side. This should be treated as a study guide and not a mock exam. More care will be taken to make sure the real exam is manageable both in difficulty and amount of time required to complete.

Problem 1. For the following sets determine whether they are (i) open, (ii) closed, and/or (iii) compact. Note that multiple or none of the properties may hold in some cases. The sets will be listed in the form $Y \subseteq X$ and you should answer for the set Y viewed as a subset of X.

(i) $\mathbb{Q} \subseteq \mathbb{R}$

(ii) $[0, 1] \subseteq [0, 1] \cup [2, 3]$

(iii) $(0, 1) \subseteq \mathbb{R}$

(iv) $\mathbb{N} = \{1, 2, 3, 4, \dots\} \subseteq \mathbb{R}$

Problem 2. Let X be a metric space.

(i) Show that the singleton set $\{x\}$ is closed for all $x \in X$.

(ii) Does this imply that all finite subsets of X are closed?

Problem 3. Show that if K and V are compact then $K \cup V$ is compact.

Problem 4. Given two sequences $(t_n)_n$ and $(s_n)_n$ of real numbers, is it true that

$$\limsup_n (s_n t_n) = \limsup_n s_n \cdot \limsup_n t_n?$$

Problem 5. Let

$$s_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n}.$$

(i) Show that $s_1 > s_3 > s_5 > s_7 > \cdots > 0$. In other words, show that the odd terms are all positive and form a decreasing subsequence. Conclude that $\lim_n s_{2n+1}$ exists.

(ii) Show that $\lim_n (s_n - s_{n+1}) = 0$.

(iii) Use (i) and (ii) to argue that $(s_n)_n$ converges.

Problem 6. Is the intersection of two connected sets connected? Prove or give a counterexample.

Problem 7. Let $(x_n)_n$ be a sequence in a metric space X. Define the following set

$$E = \{x \in X : \exists \text{a subsequence } (x_{n_k})_k \text{ of } (x_n)_n \text{ with } x_{n_k} \to x\}.$$

Show that E is closed. [Hint: Given a sequence $(y_n)_n$ in E with $y_n \to y$, build a subsequence $(x_{n_k})_k$ of $(x_n)_n$ with the property that $d(x_{n_k}, y_k) < 1/k$ for all k and show that $x_{n_k} \to y$.]