Midterm 1

The exam will last 90 minutes. Do not begin until instructed.

Please write your answers legibly in the space provided under each question, crossing out any work you do not want graded. Extra paper and/or scratch paper may be provided upon request.

Name: ______________________________

SID: ______________________________

Problem 1. Show that $\mathbb{N} \times \mathbb{R}$ is uncountable. (You may use facts about the countability or uncountablity of sets we've seen in class without proof).

Problem 2. Show that there cannot be an uncountable collection of disjoint, non-empty open intervals in $\mathbb{R}$. [Hint: Think about the density of $\mathbb{Q}$.]

Problem 3. For the following sets determine whether they are (i) open, (ii) closed, and/or (iii) compact. Note that multiple or none of the properties may hold in some cases. The sets will be listed in the form $Y \subseteq X$ and you should answer for the set Y viewed as a subset of X.

(i) $[-100, 100] \times [3, 4] \subseteq \mathbb{R}^2$

(ii) $(0, 1) \cup \{3\} \subseteq \mathbb{R}$

(iii) $\bigcup_{n \in \mathbb{N}} [n, n + 1/2] \subseteq \mathbb{R}$

Problem 4. Let X be a metric space with the following property: There exists a constant $M > 0$ such that for any two distinct points $x, y \in X$, $x \neq y$, we have that $d(x, y) \geq M$.

(i) Show that for every $x \in X$, the singleton set $\{x\}$ is open.

(ii) Deduce that every subset of X is open.

(iii) Deduce that every subset of X is closed.

Problem 5. Give an example of two sequences of real numbers $(s_n)_n$ and $(t_n)_n$ such that $(s_n t_n)_n$ converges, but neither $(s_n)_n$ nor $(t_n)_n$ converges.

This page is scratch paper. Do not write answers you want graded here unless you explicitly denote on the designated problem page that there is extra work here. In this case, you should clearly separate it from scratch work.