Final Exam Practice Questions

Note: This is to help study for the Final Exam. While the questions are similar to what a real exam may contain they may lean on the harder side. This should be treated as a study guide and not a mock exam. More care will be taken to make sure the real exam is manageable both in difficulty and amount of time required to complete.

Problem 1. For the following sets determine whether they are (i) open, (ii) closed, and/or (iii) compact. Note that multiple or none of the properties may hold in some cases. The sets will be listed in the form $Y \subseteq X$ and you should answer for the set Y viewed as a subset of X.

(i) $S^1 \setminus \{(1,0)\} \subset \mathbb{R}^2$ where $S^1 = \{v \in \mathbb{R}^2 : \|v\| = 1\}$ is the unit circle.

(ii) $\{x \in \mathbb{Q} : x^2 < 2\} \subseteq \mathbb{Q}$

Solution. (i) It is not open since no ball around any point is fully contained in S^1, let alone $S^1 \setminus \{(1,0)\}$.

It is not closed since $(1,0)$ is a limit point of $S^1 \setminus \{(1,0)\}$.

It is not compact since it is not closed.

(ii) Set $S = \{x \in \mathbb{Q} : x^2 < 2\}$. Consider the continuous function $f(x) = x^2$ from $\mathbb{Q} \to \mathbb{R}$. Then

$$S = f^{-1}((-\infty, 2))$$

and hence is open. Moreover, since $\sqrt{2}$ is irrational we also have that

$$S = f^{-1}((-\infty, 2])$$

hence is closed.

S is not compact since when viewed as a subset of $\mathbb{R}$, S is not closed (e.g. $\sqrt{2}$ is a limit point not contained in S). □

Problem 2. Let K be a compact metric space, $S \subseteq K$ any subset, and $f : S \to Y$ a uniformly continuous function.

(i) Show that for any $\delta > 0$, S may be written as the union of finitely many subsets $\{E_1, \ldots, E_n\}$ with

$$\operatorname{diam} E_i = \sup_{x,y \in E_i} d_S(x,y) < \delta$$

for all $i = 1, \ldots, n$.

(ii) Deduce using (i) that the image of f is bounded.

Solution. (i) For every $\delta > 0$ we have that $\{B_{\delta/4}(x) : x \in K\}$ is an open cover of K. By compactness, we may write

$$K = B_{\delta/4}(x_1) \cup \cdots \cup B_{\delta/4}(x_n).$$

Set $E_i = B_{\delta/4}(x_i) \cap S$. Then $S = E_1 \cup \cdots \cup E_n$ and

$$\operatorname{diam} E_i \leq \operatorname{diam} B_{\delta/4}(x_i) \leq \delta/2 < \delta$$

as required.

(ii) Since f is uniformly continuous, let $\delta > 0$ be such that

$$d_S(x, y) < \delta \implies d_Y(f(x), f(y)) < 1. \tag{$\dagger$}$$

By (i), we may write $S = E_1 \cup \cdots \cup E_n$ where each E_i has $\operatorname{diam} E_i < \delta$. By ($\dagger$) we have that

$$\operatorname{diam} f(E_i) = \sup_{x,y \in E_i} d_Y(f(x), f(y)) \leq 1.$$

Thus $f(S) = f(E_1) \cup \cdots \cup f(E_n)$ is a finite union of bounded sets, hence bounded. □

Problem 3. Compute the following integral justifying rigorously your steps:

$$\int_0^{1/2} \frac{x}{(1-x^2)^2} \, \mathrm{d}x.$$

Solution. We have an anti-derivative

$$F(x) = \frac{1}{2(1-x^2)}$$

and the integrand is continuous. Thus the fundamental theorem of calculus applies so we have

$$\int_0^{1/2} \frac{x}{(1-x^2)^2} \, \mathrm{d}x = F(1/2) - F(0) = \frac{1}{6}.$$

Alternatively, if one doesn't immediately see the anti-derivative, a more methodical approach is to use the substitution $u = 1 - x^2$. □

Problem 4. Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$ which takes the following values:

x	$f(x)$
1	7
2	3
3	5

Is it true that $f'(x)$ must equal zero at some point? Prove or give a counterexample.

Solution. By the mean value theorem, there exists some $x_1 \in (1, 2)$ such that

$$f'(x_1) = \frac{3-7}{2-1} = -4$$

and some $x_2 \in (2, 3)$ such that

$$f'(x_2) = \frac{5-3}{3-2} = 2.$$

Since derivatives satisfy the intermediate value theorem (regardless of whether they are continuous or not) there must exist some $\xi \in (x_1, x_2)$ such that $f'(\xi) = 0$. □

Problem 5. Find a power series representation of the function

$$f(x) = \frac{1}{(1-x)^2}$$

on the ball $B_1(0)$ and show that it converges to f. [Hint: Start with the geometric series.]

Solution. We have that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

on $B_1(0)$. Then we have as a result in class that power series may be differentiated term-wise on their discs of convergence. Thus

$$\frac{1}{(1-x)^2} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} = \sum_{n=0}^{\infty} nx^{n-1}.$$ □

Problem 6. Let $f : (a, b) \to \mathbb{R}$ be a twice differentiable function with f'' continuous. Prove that

$$\lim_{h\to 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} = f''(x).$$

Solution. Define a new function

$$g(h) = f(x-h) - 2f(x) + f(x+h)$$

which is twice differentiable as f is and g'' is continuous since f'' is. Notice that

$$g(0) = 0, \quad g'(0) = 0, \quad g''(0) = 2f''(x).$$

By Taylor's theorem, for every $h \neq 0$, we may find ξ be between 0 and h such that

$$\begin{aligned} g(h) &= g(0) + g'(0)h + \frac{g''(\xi)}{2}h^2 \\ &= \frac{g''(\xi)}{2}h^2 \end{aligned}$$

so that

$$\frac{g(h)}{h^2} = \frac{g''(\xi)}{2}.$$

Since g'' is continuous, letting $h \to 0$ we see that $\xi \to 0$ so

$$\lim_{h\to 0} \frac{g(h)}{h^2} = \frac{g''(0)}{2} = f''(x)$$

as desired. □

Problem 7. Let $A \subseteq [a, b]$ be a dense subset. Suppose that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are integrable functions such that $f(x) = g(x)$ for all $x \in A$. Show that

$$\int_a^b f \, dx = \int_a^b g \, dx.$$

Solution. Consider $h = f - g$ which is integrable and satisfies $h(x) = 0$ for all $x \in A$. Fix $\varepsilon > 0$. We may find a partition $P = \{x_0, \ldots, x_n\}$ with $U(P, h) - L(P, h) < \varepsilon$. By density of A, we may find $t_i \in [x_{i-1}, x_i] \cap A$ for all $i = 1, \ldots, n$. Then

$$\sum_{i=1}^{n} h(t_i)\Delta x_i = 0$$

but

$$\left| \int_a^b h \, dx \right| = \left| \int_a^b h \, dx - \sum_{i=1}^{n} h(t_i)\Delta x_i \right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we find that $\int_a^b h \, dx = 0$ which gives the result. □

Problem 8. Consider the sequence of functions $f_n : [0, 1] \to \mathbb{R}$ given by

$$f_n(x) = \begin{cases} n & \text{if } 0 < x < 1/n \\ 0 & \text{otherwise.} \end{cases}$$

Find a function f such that $f_n \to f$ pointwise. Is the convergence uniform?

Solution. Given any $0 < x \leq 1$ for n sufficiently large we have that $1/n < x$ so $f_n(x) = 0$. Thus $f_n(x) \to 0$ in this case. Moreover, we have that $f_n(0) = 0$ for all n so $f_n(0) \to 0$.

We conclude that $f_n \to 0$ pointwise. This convergence is not uniform, however. Indeed, there exists no N such that $|f_n(x)| < 1$ for all $x \in [0, 1]$ and $n \geq N$ since $\sup |f_n(x)| = n$. □