Final Exam

The exam will last 120 minutes. Do not begin until instructed.

Please write your answers legibly in the space provided under each question, crossing out any work you do not want graded. Extra paper and/or scratch paper may be provided upon request.

Name: ____

SID: ____

Problem 1. For the following sets determine whether they are (i) open, (ii) closed, and/or (iii) compact. Note that multiple or none of the properties may hold in some cases. The sets will be listed in the form $Y \subseteq X$ and you should answer for the set Y viewed as a subset of X.

(i) $\{v \in \mathbb{R}^2 : 1 \leq \|v\| \leq 2\} \subseteq \mathbb{R}^2$

(ii) $\{1\} \subseteq \{1\} \cup (2, 3)$

(iii) $\{x \in \mathbb{Q} : 0 \leq x < 2\} \subseteq \mathbb{Q}$.

Problem 2. Let X be a metric space and let $E \subseteq X$ be closed and $U \subseteq X$ be open. Show that

$$E \setminus U = \{x \in E : x \notin U\}$$

is closed in X.

Problem 3. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function with $f(a) = 0$ such that $|f'(x)| \leq A|f(x)|$ for all x, for some $A \in \mathbb{R}$. Show that $f(x) = 0$ via the following steps.

(i) Fix some $x_0 \in (a, b)$ and set

$$M_0 = \sup_{x \in [a, x_0]} |f(x)|, \quad M_1 = \sup_{x \in [a, x_0]} |f'(x)|.$$

Show that for $a \leq x \leq x_0$ we have

$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0.$$

(ii) Deduce from (i) that if $A(x_0 - a) < 1$ then $f = 0$ on $[a, x_0]$.

(iii) Deduce that $f = 0$ on $[a, b]$.

Problem 4. Let

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} nx \sin(2\pi n x^2).$$

(i) Show that f defines a continuous function $\mathbb{R} \to \mathbb{R}$.

(ii) Compute

$$\int_0^1 f \, \mathrm{d}x$$

making sure to justify your computations.

[Hint: Weierstrass M-test. You may use any basic facts about trig functions that you wish without proof, in particular that $\cos'(x) = -\sin(x)$, $\sin'(x) = \cos(x)$ and $\cos(2\pi m) = 1$ for all $m \in \mathbb{Z}$.]

Problem 5. Suppose that $(f_n)_n$ is a sequence of uniformly continuous functions $X \to Y$. If $f_n \to f$ uniformly, then show that f is uniformly continuous.

Problem 6. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous. Is it true that

$$\int_0^{\pi} f(\sin x) \cos x \, \mathrm{d}x = 0?$$

Prove or give a counterexample. [You may use basic facts about trig functions without proof.]

This page is scratch paper. Do not write answers you want graded here unless you explicitly denote on the designated problem page that there is extra work here. In this case, you should clearly separate it from scratch work.