Polynomial Wolff Axioms and Multilinear Kakeya-type Estimates for Bent Tubes in $\mathbb{R}^n$

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August 2018

Abstract

In this paper we consider the applicability of Guth and Zahl's polynomial Wolff axioms to bent tubes. We demonstrate that Guth and Zahl's multilinear bounds hold for tubes defined by low degree algebraic curves with bounded C^2-norms. To show this we give an exposition of their proof in a n-dimensional, k-linear context.

In considering the ability to obtain linear bounds using the multilinear bounds we utilize the strategy of Guth and Bourgain. We find that the multilinear bounds obtained from Guth and Zahl's technique break the inductive structure of this process and thus provide inferior bounds to the endpoint cases of Bennett, Carbery, and Tao's multilinear bounds. We discuss future research directions, which could eventually remedy this, that improve multilinear bounds by adding the assumption that the collection of tubes lie near a k-plane.

1 Introduction

In this paper we consider recent techniques relating to the Kakeya conjecture and their applicability to bent tubes. The Kakeya conjecture in its maximal function form deals with the Kakeya maximal operator: Let $f \in L_{1,\text{loc}}(\mathbb{R}^n)$, then define the operator $\mathcal{M}_\delta$ by

$$\mathcal{M}_\delta f : \mathbb{R}^n \to \mathbb{R}, \qquad \mathcal{M}_\delta f(e) = \sup_{T//e} \frac{1}{|T|} \int_T |f|,$$

where T ranges over cylinders (or tubes) of length 1, cross sectional radius δ, and axis parallel to e, and $|\cdot|$ denotes the Lebesgue measure. It is conjectured that for every $\epsilon > 0$ there is a constant C_ϵ dependent on ϵ such that

$$\|\mathcal{M}_\delta f\|_n \leq C_\epsilon \delta^{-\epsilon} \|f\|_n.$$

Standard interpolation of the conjectured bound with the trivial L^∞ bound gives the equivalent conjecture

$$\|\mathcal{M}_\delta f\|_q \leq C_\epsilon \delta^{-(n/p-1+\epsilon)} \|f\|_p, \qquad 1 \leq p \leq n,\ q = (n-1)p', \tag{1}$$

where p' is the dual exponent to p. This puts the conjecture into a form that opens the door to a whole family of partial results. The most significant of which has been the work of Wolff that showed (1) to hold for $p = (n+2)/2$ (see [8]). A good survey of results relating to this conjecture was given by Katz and Tao and can be found in [6].

While the above gives the maximal function formulation of the Kakeya conjecture, a very common, discretized formulation deals with the overlap of tubes (neighborhoods of line segments). Suppose that $\mathbb{T}$ is a set of tubes of length 1 and radius δ in $\mathbb{R}^n$ that are δ-separated. By δ-separated, it is meant that for each pair of tubes in $\mathbb{T}$, their axes have an angular separation of at least δ. It is then conjectured that for every $1 \leq p \leq n$ and $\epsilon > 0$ we have that

$$\Big\| \sum_{T \in \mathbb{T}} \chi_T \Big\|_{p/(p-1)} \leq C_\epsilon \delta^{-(n/p-1+\epsilon)}. \tag{2}$$

The reason that this conjecture is also referred to as the Kakeya maximal function conjecture lies in that (2) is dual to the estimate (1). Note that by Hölder, (2) implies the weaker bound

$$\Big| \bigcup_{T \in \mathbb{T}} T \Big| \geq C_\epsilon \delta^{n-p+\epsilon}$$

which is also of interest ($|\cdot|$ again denotes the Lebesgue measure).

In proving that (2) holds for $p = (n+2)/2$, Wolff utilized a consequence of the δ-separation of the tubes that roughly states that not too many tubes can lie too close to a plane. Recently, however, Guth and Zahl (see [5]) gave the first improvement on Wolff's partial result in $\mathbb{R}^4$ by considering a generalization of Wolff's property. Guth and Zahl considered tubes satisfying the polynomial Wolff axioms, which in short says that not too many tubes can lie to close to a low degree algebraic variety. By doing this they proved that collections of tubes satisfying the polynomial Wolff axioms satisfied (2) for $p = 3 + 1/28$ (compared to Wolff's result of $p = 3$), and it was subsequently proven in [7] by Katz and Rogers that collections of tubes that are δ-separated indeed satisfy the polynomial Wolff axioms. In their proof, Guth and Zahl also make use of multilinear bounds. Roughly speaking, multilinear bounds are bounds founded on the addition assumption that for every x, most k-tuples of tubes going through x point in k quantitatively linearly independent directions (such an estimate would be referred to as a k-linear estimate).

In this paper we consider the polynomial Wolff axioms in relation to bent tubes. In the traditional discretized form of the Kakeya conjecture, one considers the size of the overlap of straight tubes. It is well known that this hypothesis is necessary for the full Kakeya conjecture, but the exact function of the straightness hypothesis is yet to be perfectly understood. Any successful proof of the Kakeya conjecture will have to exploit straightness, making it useful to understand its function. The results in this paper are to serve as a bellwether to where straightness of the tubes is necessary, specifically in the context of the polynomial Wolff axioms and multilinear estimates.

In demonstrating the agnosticism of Guth and Zahl's techniques a generalization of Guth and Zahl's multilinear bound from [5] is proven. Roughly speaking, if a set $\mathbb{T}$ of bent tubes in $\mathbb{R}^n$ satisfies the polynomial Wolff axioms (i.e. are never too concentrated on a low degree algebraic variety) and most tubes through a given point point in k linearly independent directions, then

$$\Big\| \sum_{T\in\mathbb{T}} \chi_T \Big\|_{\frac{nk+n-k}{n-1}} \leq C_\epsilon \delta^{n-\frac{n-k}{k}-\epsilon} |\mathbb{T}|^{\frac{n}{n-1}}.$$

This result is stated more precisely in Theorem 2.6. To prove this, Guth and Zahl's technique divides the domain of interest into a covering by low degree algebraic varities. By choosing this covering appropriately, it follows from the polynomial Wolff axioms that the tubes must be sufficiently evenly distributed over the algebraic varities, leading to improved bounds.

2 Definitions and Main Results

We begin by introducing the following notation that is used throughout the paper.

Definition 2.1. Let f and g be real valued functions, then $f \lesssim g$ means that there exists a constant C such that $f \leq Cg$. Writing $f \sim g$ means that $f \lesssim g$ and $f \gtrsim g$. Note that what the constant C depends on will be clear by context, but throughout the paper C will never depend on properties relating to the collection of tubes.

Definition 2.2. Let f and g be real valued functions, then $f \lessapprox g$ means that for all $\epsilon > 0$ there exists a constant C_ϵ (dependent on ϵ) such that $f \leq C_\epsilon \delta^{-\epsilon} g$ (note that here δ represents a variable that f and g will often be functions dependent on). Similarly, writing $f \approx g$ means that $f \lessapprox g$ and $f \gtrapprox g$.

Using this notation, (2), for example, can be rewritten as

$$\Big\| \sum_{T\in\mathbb{T}} \chi_T \Big\|_{p/(p-1)} \lessapprox \delta^{-(n/p-1)}.$$

Since the results of this paper focus on bent tubes, we also define what will be understood by a δ-tube in the context of this paper.

Definition 2.3. A *δ-tube* is the δ-neighborhood of an algebraic curve of degree $\lesssim 1$ with C^2-norm $\lesssim 1$.

Remark 2.1. By Definition 2.1, the statement "algebraic curve of degree $\lesssim 1$" may be thought of as "algebraic curve of bounded degree". The important difference, however, is that $\lesssim 1$ represents a fixed implicit constant in the definition whereas one could have a set of algebraic curves all of bounded degree, but whose degrees are not bounded by a single fixed constant.

Specifically, in this paper we consider Guth and Zahl's polynomial Wolff axioms and its applications to bent tubes in dimensions above $\mathbb{R}^4$. The standard Wolff axioms require that not too many tubes lie close to a plane, while the polynomial Wolff axioms, first introduced in [5], require that not too many tubes lie close to low degree algebraic varieties. Assuming this constraint only, Guth and Zahl managed to show that for collections of tubes $\mathbb{T}$ in $\mathbb{R}^4$ of size $\sim \delta^{-3}$

$$\Big| \bigcup_{T\in\mathbb{T}} T \Big| \gtrapprox \delta^{1-1/28}. \tag{3}$$

As it turns out, however, many of the techniques featured in [5] appear agnostic when it comes to tubes centered around low degree (bent) algebraic curves over degree one (straight) algebraic curves. Technically, the polynomial Wolff axioms say that not too many tubes can intersect a semi-algebraic set with a high enough density.

Definition 2.4 (Semi-algebraic set)**.** A *semi-algebraic set* is a set of the form

$$S = \{x \in \mathbb{R}^n \ : \ f_1(x) \rhd_1 0, f_2(x) \rhd_2 0, \dots, f_i(x) \rhd_i 0\}$$

where $\rhd_j \in \{=, >\}$ and $f_1, \dots, f_i$ are polynomials in x. The *complexity* of S is then the minimum value of $\deg(f_1) + \cdots + \deg(f_i)$ over all representations of S of the above form.

Definition 2.5 (Polynomial Wolff axioms)**.** A collection $\mathbb{T}$ of δ-tubes is said to satisfy the *polynomial Wolff axioms* if for every semi-algebraic set S of complexity $\leq E$ and every $\delta \leq \lambda \leq 1$ we have that

$$\#\{T \in \mathbb{T} \ : \ |T \cap S| \geq \lambda |T|\} \leq K_E |S| \delta^{1-n} \lambda^{-n},$$

where K_E is some constant dependent only on E, and $|\cdot|$ denotes the Lebesgue measure.

At the heart of Guth and Zahl's proof of (3) is an improved trilinear bound that is obtained through emerging polynomial methods, first introduced by Dvir in [3] to resolve Wolff's modification of the Kakeya problem for finite fields, in conjunction with the polynomial Wolff axioms. In short, by decomposing the collection of tubes into a union of low degree algebraic varieties we get a dichotomy: Either the tubes are concentrated on an algebraic variety, or well distributed over them. The hypothesis that the tubes satisfy the polynomial Wolff axioms eliminate the former possibly, leading to improved multilinear bounds. In this paper, we extend Guth and Zahl's multilinear bounds to k-linear bounds in $\mathbb{R}^n$ for bent tubes. Specifically, we conclude the following theorem:

Theorem 2.6. *Assume $2 \leq k \leq n-1$. Let $\mathbb{T}_1, \dots, \mathbb{T}_k$ be sets of δ-tubes in $\mathbb{R}^n$ that satisfy the polynomial Wolff axioms. Then,*

$$\int_{B(0,2)} \Big(\sum_{(T_1,\dots,T_k) \in \mathbb{T}_1 \times \cdots \times \mathbb{T}_k} \chi_{T_1} \cdots \chi_{T_k} |v_1 \wedge \cdots \wedge v_k|^{\frac{kn}{nk-k+n}} \Big)^{\frac{nk+n-k}{k(n-1)}} \lessapprox \delta^{n-\frac{n-k}{k}} \Big(\prod_{i=1}^{k} |\mathbb{T}_i| \Big)^{\frac{n}{k(n-1)}}.$$

Remark 2.2. In the above theorem, $|v_1 \wedge \cdots \wedge v_k|$ represents the Lebesgue measure of the parallelpiped spanned by $v_1, \dots, v_k$.

3 Multilinear Bounds

The best known general multilinear bounds are due to Guth and Bourgain in [2]. Although they only prove the trilinear bound in $\mathbb{R}^4$, their proof implies the following k-linear bounds in $\mathbb{R}^n$.

Theorem 3.1 (Bourgain-Guth, [2])**.** *Assume $2 \leq k \leq n-1$. Let $\mathbb{T}_1, \dots, \mathbb{T}_k$ be sets of δ-tubes in $\mathbb{R}^n$. Then there exists a constant C independent of the set of tubes such that*

$$\int_{B(0,2)} \Big(\sum_{(T_1,\dots,T_k) \in \mathbb{T}_1 \times \cdots \times \mathbb{T}_k} \chi_{T_1} \cdots \chi_{T_k} |v_1 \wedge \cdots \wedge v_k| \Big)^{\frac{1}{k-1}} \leq C \prod_{i=1}^{k} \delta^{n/k} |\mathbb{T}_i|^{1/(k-1)}.$$

This estimate is the endpoint case of the multilinear estimates first proven by Bennett, Carbery, and Tao in [1]. The latter states that if $\mathbb{T}$ is a set of δ-tubes in $\mathbb{R}^n$, with $|\mathbb{T}| \sim \delta^{1-n}$, such that most k-tuples of tubes point in k linearly independent directions then

$$\Big| \bigcup_{T \in \mathbb{T}} T \Big| \gtrapprox \delta^{n-k}. \tag{4}$$

The above estimates require no constraints on the sets of tubes. However, as Guth and Zahl found, these bounds admit the following improvement in $\mathbb{R}^4$ when the additional assumption of the polynomial Wolff axioms is added.

Theorem 3.2 (Guth-Zahl, [5])**.** *Let $\mathbb{T}_1, \ldots, \mathbb{T}_k$ be sets of δ-tubes in $\mathbb{R}^4$ that satisfy the polynomial Wolff axioms. Then we have*

$$\int_{B(0,2)} \Big(\sum_{(T_1,\ldots,T_k)\in\mathbb{T}_1\times\cdots\times\mathbb{T}_k} \chi_{T_1}\cdots\chi_{T_k} |v_1 \wedge \cdots \wedge v_k|^{12/13} \Big)^{13/9} \lessapprox \delta^{11/3} \Big(\prod_{i=1}^{3} |\mathbb{T}_i| \Big)^{4/9}.$$

More general of course is Theorem 2.6 which tells us that if $\mathbb{T}$ is a set of δ-tubes, with $|\mathbb{T}| \sim \delta^{1-n}$, such that most k-tuples of tubes point in k linearly independent directions then

$$\Big| \bigcup_{T\in\mathbb{T}} T \Big| \gtrapprox \delta^{\frac{n-1}{n}\cdot(n-k)}.$$

This gives a clear improvement over (4). The proof of Theorem 2.6 in the case of $k=3$ and $n=4$ is given by Guth and Zahl in [5], but their method extends to k-linear in $\mathbb{R}^n$. Moreover, their proof allows the result to be rephrased in a manner that takes in existing bounds and returns an improved bound.

Proposition 3.3. *Let $\mathcal{K}_{k,n}(p,q,e_1,e_2)$ be the proposition that*

$$\int_{B(0,2)} \Big(\sum_{(T_1,\ldots,T_k)\in\mathbb{T}_1\times\cdots\times\mathbb{T}_k} \chi_{T_1}\cdots\chi_{T_k} |v_1 \wedge \cdots \wedge v_k|^{q} \Big)^{p/k} \lessapprox \delta^{e_1} \Big(\prod_{i=1}^{k} |\mathbb{T}_i| \Big)^{e_2/k}, \tag{5}$$

whenever $\mathbb{T}_1, \ldots, \mathbb{T}_k$ are sets of δ-tubes in $\mathbb{R}^n$ satisfying the polynomial Wolff axioms. Then

$$\mathcal{K}_{k,n}(p,q,e_1,e_2) \Rightarrow \mathcal{K}_{k,n}(p',q',e_1',e_2')$$

where

$$\begin{aligned} p' &= 1 + \frac{n}{n-1}\cdot\frac{p-1}{e_2} \\ q' &= \frac{qp}{p-1+e_2(n-1)/n} \\ e_1' &= 1 + \frac{n}{n-1}\cdot\frac{e_1-1}{e_2} \\ e_2' &= \frac{n}{n-1}. \end{aligned}$$

Corollary 3.4. *Theorem 2.6 holds.*

Proof. By Theorem 4, we have that $\mathcal{K}_{k,n}\big(\frac{k}{k-1}, 1, n, \frac{k}{k-1}\big)$ holds with no assumptions on the tubes. Assuming that $\mathbb{T}_1, \ldots, \mathbb{T}_k$ satisfy the polynomial Wolff axioms, we can then apply Proposition 3.3. $\square$

A major tool utilized by Guth and Zahl are grains and grains decompositions for which we restate the definition given in [5].

Definition 3.5 (Grain)**.** A *grain* of complexity D is the δ-neighborhood of a semi-algebraic set in $\mathbb{R}^n$ of complexity D and dimension $\leq n-1$.

Definition 3.6 (Grains decomposition)**.** A *grains decomposition* of complexity D and error ϵ of a set of δ-cubes $\mathcal{Q}$ is a set $\mathcal{G}$ of grains of complexity $\leq D$ such that the following holds: There exists a collection of subsets of $\mathcal{Q}$, $\{\mathcal{Q}_G\}_{G\in\mathcal{G}}$, such that for each $G \in \mathcal{G}$, we have that for all $Q \in \mathcal{Q}_G$, $Q \subset G$, the sets $\{\mathcal{Q}_G\}_{G\in\mathcal{G}}$ are disjoint, and the following three properties hold:

- Roughly every cube is accounted for, i.e.

$$\sum_{G\in\mathcal{G}} |\mathcal{Q}_G| \gtrapprox |\mathcal{Q}|. \tag{6}$$

- The grains are evenly distributed, i.e.

$$|\mathcal{Q}_G| \approx |\mathcal{Q}|/|\mathcal{G}|. \tag{7}$$

- A tube doesn't intersect too many grains, i.e. for every δ-tube T

$$\#\{G \in \mathcal{G} \,:\, T \cap G \neq \emptyset\} \lessapprox |\mathcal{G}|^{1/n}. \tag{8}$$

The primary function of grains decompositions comes in one's ability to invoke an arbitrarily precise grains decomposition at the expense of its complexity. For this we restate Proposition 3.2 of [5] and an overview of its proof with justification that it holds for bent tubes.

Proposition 3.7. *For every set of δ-cubes $\mathcal{Q}$ in $\mathbb{R}^n$ with $|\mathcal{Q}| = \mathcal{O}(\delta^m)$ and $\epsilon > 0$ there exists a grains decomposition of $\mathcal{Q}$ with complexity $D(\epsilon, m, n)$ and error ϵ.*

Proof. The proof of this proposition makes heavy use of the polynomial partitioning theorem introduced in [4], which for any open set $U \subset \mathbb{R}^n$ gives the existence of a of a degree D algebraic variety that divides U into $\sim D^n$ disjoint, connected, equal volume subsets called cells. Consider $U = \cup_{Q\in\mathcal{Q}} Q$, the proof utilizes this partitioning theorem to iteratively divide U and subsequent cells into smaller cells. At each iteration we can consider the δ-neighborhood of these cell walls. Eventually the process will terminate when a fraction $\gtrapprox 1$ of the δ-cubes lie in the neighborhood of one of these cell walls, which become our grains.

The only modification necessary to account for bent tubes is in showing that (8) still holds under this construction despite the generalization to bent tubes. Following the proof of Proposition 3.2 in [5] we know that the process will terminate in

$$s \lesssim \log_{D^n} |\mathcal{Q}| \tag{9}$$

steps. Since a δ-tube is an algebraic curve of degree $\leq d$ we know that a tube will intersect at most $d(D+1)$ cells on each iteration. Thus we have that T intersects $\leq d^s(D+1)^s \lessapprox d^s|\mathcal{G}|^{1/n}$ grains. Using (9) we have

$$d^s \lesssim |\mathcal{Q}|^{\log_{D^n} d} \lesssim \delta^{-m \log_{D^n} d}$$

Since $d \lesssim 1$, by choosing $D = D(\epsilon, m, n)$ sufficiently large we see that the term d^s term can be absorbed into the $\delta^{-\epsilon}$ term. $\square$

Proof of Proposition 3.3. Fix k and n and assume $\mathcal{K}_{k,n}(p, q, e_1, e_2)$ holds. We are considering the integral

$$\int_{B(0,2)} \Big(\sum_{(T_1,\ldots,T_k)\in\mathbb{T}_1\times\cdots\times\mathbb{T}_k} \chi_{T_1}\cdots\chi_{T_k}|v_1\wedge\cdots\wedge v_k|^{q'}\Big)^{p'/k} \tag{10}$$

where q' and p' are as stated in Proposition 3.3. In this proof, replace each tube with the union of all δ-tubes intersecting it. Doing this allows for convenience of notation, but has no effect on the results since these sets are contained in the 2δ-neighborhoods of the central curves of each tube. Firstly, dyadic pigeonhole $|v_1 \wedge\cdots\wedge v_k|$ to find a dyadic θ such that

$$(10) \sim \int_{B(0,2)} \Big(\sum_{\substack{(T_1,\ldots,T_k)\in\mathbb{T}_1\times\cdots\times\mathbb{T}_k \\ |v_1\wedge\cdots\wedge v_k|\sim\theta}} \chi_{T_1}\cdots\chi_{T_k}|v_1\wedge\cdots\wedge v_k|^{q'}\Big)^{p'/k}.$$

Now let $\mathcal{Q}$ be the set of all δ-cubes intersecting $B(0,2)$, so that

$$\int_{B(0,2)} \Big(\sum_{\substack{T_1,\ldots,T_k \\ |v_1\wedge\cdots\wedge v_k|\sim\theta}} \chi_{T_1}\cdots\chi_{T_k}|v_1\wedge\cdots\wedge v_k|^{q'}\Big)^{p'/k} = \int_{\bigcup_{Q\in\mathcal{Q}} Q} \Big(\sum_{\substack{T_1,\ldots,T_k \\ |v_1\wedge\cdots\wedge v_k|\sim\theta}} \chi_{T_1}\cdots\chi_{T_k}|v_1\wedge\cdots\wedge v_k|^{q'}\Big)^{p'/k}.$$

Dyadic pigeonhole the contribution of each cube to the integral to refine the set $\mathcal{Q}$ so that if $A = \bigcup_{Q \in \mathcal{Q}} Q$ then

$$\int_{B(0,2)} \Big(\sum_{\substack{T_1,\dots,T_k \\ |v_1 \wedge \cdots \wedge v_k| \sim \theta}} \chi_{T_1} \cdots \chi_{T_k} |v_1 \wedge \cdots \wedge v_k|^{q'} \Big)^{p'/k} \sim \int_A \Big(\sum_{\substack{T_1,\dots,T_k \\ |v_1 \wedge \cdots \wedge v_k| \sim \theta}} \chi_{T_1} \cdots \chi_{T_k} |v_1 \wedge \cdots \wedge v_k|^{q'} \Big)^{p'/k}$$

where each cube in $\mathcal{Q}$ contributes roughly evenly to the integral on the RHS. Invoke Proposition 3.7 to find a grains decomposition $\mathcal{G}$ of $\mathcal{Q}$.

Let $\mathcal{G}$ be our grains decomposition of $\mathcal{Q}$. We have that

$$\int_{B(0,2)} \Big(\sum_{\substack{T_1,\dots,T_k \\ |v_1 \wedge \cdots \wedge v_k| \sim \theta}} \chi_{T_1} \cdots \chi_{T_k} \theta^{q'} \Big)^{p'/k} \lessapprox \sum_{G \in \mathcal{G}} \int_{A \cap G} \Big(\sum_{\substack{T_1,\dots,T_k \\ |v_1 \wedge \cdots \wedge v_k| \sim \theta}} \chi_{T_1} \cdots \chi_{T_k} \theta^{q'} \Big)^{p'/k}.$$

Each tube intersecting a grain will intersect for some length l along its central curve. Let $\mathrm{CC}(S, x)$ mean the connected component of the set S containing x. The length of the intersection of a grain G and a tube T containing the point x is given by $\mathrm{diam}(\mathrm{CC}(G \cap T, x))$. Dyadic pigeonhole these lengths to find $l_1, \dots, l_k$ such that

$$(10) \lessapprox \sum_{G \in \mathcal{G}} \int_{A \cap G} \Big(\sum_{\substack{(T_1,\dots,T_k) \in \mathbb{T}_1 \times \cdots \times \mathbb{T}_k \\ |v_1 \wedge \cdots \wedge v_k| \sim \theta \\ l_i \le \mathrm{diam}(\mathrm{CC}(T_i \cap G, x)) \le 2l_i}} \chi_{T_1} \cdots \chi_{T_k} \theta^{q'} \Big)^{p'/k}. \tag{11}$$

Assume that $l_1 = \max(l_1, \dots, l_k)$. Cover each grain by radius Cl_1 balls that are boundedly overlapping such that every subset of G of diameter $\le 2l_1$ is fully contained in one of these balls. The intersection of a grain G with one of these balls will be called a sub-grain G' with parent G. If G' is a sub-grain of G, let $\mathcal{Q}_{G'} = \{Q \in \mathcal{Q}_G \ : \ Q \subset G'\}$. Let $\mathcal{G}'$ denote the set of sub-grains and let $\mathcal{Q}' = \bigcup_{G' \in \mathcal{G}'} \mathcal{Q}_{G'}$. Note that after dyadic pigeonholing and refining the sub-grains in $\mathcal{G}'$, we can assume (6), (7), and (8) hold for $\mathcal{G}'$ (doing this makes $|\mathcal{G}'|$ smaller by a factor $\lessapprox 1$), which makes $\mathcal{G}'$ a grains decomposition of $\mathcal{Q}'$. Let $A' = \bigcup_{Q \in \mathcal{Q}'} Q$.

For $i = 1, \dots, k$ and $G' \in \mathcal{G}'$, define

$$\mathbb{T}_{i,G'} = \{T \in \mathbb{T}_i \ : \ \text{there exists a connected set } W \subset T \cap G',\ l_i \le \mathrm{diam}(W) \le 2l_i\}.$$

By construction of the sub-grains, if T is a δ-tube and G is a grain such that for some x, $l_i \le \mathrm{diam}(\mathrm{CC}(T \cap G, x)) \le 2l_i$, then $T \in \mathbb{T}_{i,G'}$ for some sub-grain G' of G. Hence, by (11) we have

$$(10) \lessapprox \theta^{q'p'/k} \sum_{G' \in \mathcal{G}'} \int_{A' \cap G'} \Big(\sum_{\substack{(T_1,\dots,T_k) \in \mathbb{T}_{1,G'} \times \cdots \times \mathbb{T}_{k,G'} \\ |v_1 \wedge \cdots \wedge v_k| \sim \theta}} \chi_{T_1} \cdots \chi_{T_k} \Big)^{p'/k}.$$

Refine the sub-grains so that each sub-grain contributes roughly equally to the integral.

Since each sub-grain is the $C\delta$-neighborhood of a semi-algebraic set of dimension $\le n-1$ contained in a ball of radius $\lesssim l_1$ we have

$$|G'| \lesssim l_1^{n-1} \delta \tag{12}$$

(see [9]; note that the result of [9] applies only to algebraic varieties, but tracing the construction of G', G' is the neighborhood of an algebraic variety intersected with a radius $\lesssim l_1$ ball).

Dyadic pigeonhole the grains to find numbers $N_1, \dots, N_k$ so that $N_i \le |\mathbb{T}_{i,G'}| \le 2N_i$ for each $G' \in \mathcal{G}'$.

Since each G' satisfies (8),

$$N_i|\mathcal{G}'| \sim \sum_{G' \in \mathcal{G}'} |\mathbb{T}_{i,G'}| \leq \sum_{T \in \mathbb{T}_i} |\{G' \in \mathcal{G}' \ : \ G' \cap T \neq \emptyset\}| \lessapprox |\mathbb{T}_i| \cdot |\mathcal{G}'|^{1/n}.$$

Hence,

$$N_i \lessapprox |\mathbb{T}_i| \cdot |\mathcal{G}'|^{-\frac{n-1}{n}}. \tag{13}$$

Dyadic pigeonhole and refine the set $\mathcal{Q}'$ and the associated sets $\mathcal{Q}_{G'}$ to find numbers μ and $\mu_1, \dots, \mu_k$ such that if $Q \in \mathcal{Q}_{G'}$ for some $G' \in \mathcal{G}'$ and $x \in Q$, then

$$\sum_{\substack{(T_1,\dots,T_k) \in \mathbb{T}_{1,G'} \times \cdots \times \mathbb{T}_{k,G'} \\ |v_1 \wedge \cdots \wedge v_k| \sim \theta}} \chi_{T_1}(x) \dots \chi_{T_k}(x) \sim \mu^k$$

and $\sim \mu_i$ tubes from $\mathbb{T}_{i,G'}$ pass through x for each $i = 1, \dots, k$. Due to all the dyadic pigeonholing,

$$(10) \lessapprox |A'| \mu^{p'} \theta^{p'q'/k}. \tag{14}$$

Notice that

$$\mu \leq \Big(\prod_{i=1}^{k} \mu_i\Big)^{1/k}. \tag{15}$$

Also notice that

$$\begin{aligned} \mu_i |A'| &\lesssim \sum_{G' \in \mathcal{G}'} \sum_{T \in \mathbb{T}_{i,G'}} |T \cap A' \cap G'| \\ &\leq \sum_{G' \in \mathcal{G}'} \sum_{T \in \mathbb{T}_{i,G'}} |T \cap G'| \\ &\leq |\mathcal{G}'| N_i l_i \delta^{n-1} \\ &\lessapprox |\mathbb{T}_i| \cdot |\mathcal{G}'|^{1/n} l_i \delta^{n-1} \end{aligned}$$

where the last line uses (13). Hence,

$$l_i \gtrapprox \mu_i |A'| \cdot |\mathcal{G}|^{-1/n} \delta^{1-n} |\mathbb{T}_i|^{-1}.$$

Since the sets of tubes $\mathbb{T}_1, \dots, \mathbb{T}_k$ satisfy the polynomial Wolff axioms, using (12) we have

$$\begin{aligned} N_i &\lesssim K l_i^{n-1} \delta \delta^{1-n} l_i^{-n} \\ &= K \delta^{2-n} l_i^{-1} \\ &\lessapprox \delta^{2-n} \mu_i^{-1} |A'|^{-1} |\mathcal{G}|^{1/n} \delta^{n-1} |\mathbb{T}_i|, \end{aligned} \tag{16}$$

where $K = \sup_{1 \leq E \leq D(\epsilon,n)} K_E$ with the K_E being as in Definition 2.5. Note that the K can be absorbed by the C_ϵ implicit constant since the complexity of the grains are $\lesssim D(\epsilon, n)$.

Refine $\mathcal{G}'$ so that we still have $|\mathcal{Q}_{G'}| \approx |\mathcal{Q}'|/|\mathcal{G}'|$ for all $G' \in \mathcal{G}'$. Let $G' \in \mathcal{G}'$. Thus

$$\begin{aligned}
|A'| \cdot |\mathcal{G}'|^{-1} \mu^p \theta^{qp/k} &\approx \int_{A' \cap G'} \Big(\sum_{\substack{(T_1,\dots,T_k) \in \mathbb{T}_{1,G'} \times \dots \times \mathbb{T}_{k,G'} \\ |v_1 \wedge \dots \wedge v_k| \sim \theta}} \chi_{T_1} \cdots \chi_{T_k} \theta^q \Big)^{p/k} \\
&\lessapprox \delta^{e_1} \Big(\prod_{i=1}^k N_i \Big)^{e_2/k} \\
&\lessapprox \delta^{e_1} \Big(\prod_{i=1}^k |\mathbb{T}_i| \cdot |\mathcal{G}'|^{-\frac{n-1}{n}} \Big)^{x/k} \Big(\prod_{i=1}^k \delta^{2-n} \mu_i^{-1} |A'|^{-1} |\mathcal{G}|^{1/n} \delta^{n-1} |\mathbb{T}_i| \Big)^{(e_2 - x)/k} \\
&= \delta^{e_1 + e_2 - x} |A'|^{-(e_2 - x)} |\mathcal{G}'|^{-1} \Big(\prod_{i=1}^k \mu_i \Big)^{-\frac{e_2 - x}{k}} \Big(\prod_{i=1}^k |\mathbb{T}_i| \Big)^{\frac{e_2}{k}}
\end{aligned} \tag{17}$$

where

$$x = \frac{n + e_2}{n}$$

is chosen such that there is cancellation of the $|\mathcal{G}'|^{-1}$ terms. The second line uses the assumption that $\mathcal{K}_{k,n}(p, q, e_1, e_2)$ holds and the third line uses (13) and (16). Using (15) we see that (17) gives

$$|A'| \mu^{p'} \theta^{p'q'/k} \lessapprox \delta^{e_1'} \Big(\prod_{i=1}^k |\mathbb{T}_i| \Big)^{e_2'}$$

where p', q', e_1', and e_2' are as given in Proposition 3.3. Referring to (14) we see that this completes the proof. □

4 Linear Bounds

We follow the technique of [2] to go from k-linear bounds to linear bounds. The general idea behind this is to evaluate

$$\int_{B(0,2)} \Big| \sum_{v \in \tau} \chi_T \Big|^p,$$

where τ is a cap of radius between δ and $\mathcal{O}(1)$. We start with a cap of radius δ and induct to get to a cap of radius $\mathcal{O}(1)$. To induct we will break τ into smaller caps of radius radius$(\tau)/K$ and use a broad-narrow decomposition. A point will be narrow if most tubes through x are contained in the (radius$(\tau)/K$)-neighborhood of a $(k-1)$-plane, and a point will be broad otherwise. We write

$$\int_{B(0,2)} \Big| \sum_{v \in \tau} \chi_T \Big|^p = \int_{\text{Broad}} \Big| \sum_{v \in \tau} \chi_T \Big|^p + \int_{\text{Narrow}} \Big| \sum_{v \in \tau} \chi_T \Big|^p.$$

The integral over the broad points is easily handled by the k-linear estimate. For the narrow points, using Holder and the fact that for each x most tubes are contributed to by $\sim K^{k-2}$ caps, the integral over the narrow points can be handled directly by the inductive hypothesis.

To handle the induction, we need to add the geometric hypothesis used in [2]. Let $\mathbb{T}$ be a set of δ-tubes. We say that the tubes in $\mathbb{T}$ are δ-separated if to each tube $T_i \in \mathbb{T}$ we can associate a vector $y_i \in S^{n-1}$ such that the vectors $\{y_i\}$ are δ-separated and for every $T_i, T_j \in \mathbb{T}$, $|v_i(x) - v_j(x)| \gtrsim |y_i - y_j|$ for all $x \in T_i \cap T_j$. When every tube is straight, this constraint is equivalent to standard definition of δ-separation, but formulating it in this manner allows us to also handle bent tubes.

Theorem 4.1. *Let $\mathbb{T}$ be a set of δ-separated δ-tubes satisfying the polynomial Wolff axioms. If*

$\mathcal{K}_{k,n}(p_0, q_0, e_1, e_2)$ *holds, then*

$$\Big\| \sum_{T\in\mathbb{T}} \chi_T \Big\|_p \lessapprox \delta^{-(n-1)+((n-1)(p_0-e_2)+e_1)/p}$$

for

$$p \geq \max\Big(p_0,\ 1 + \frac{(n-1)(p_0-e_2) + \frac{k-1}{k}p_0q_0}{n-k+1}\Big).$$

To prove this we will instead prove a stronger proposition that emits an inductive proof.

Proposition 4.2. *Let $\mathbb{T}$ be a set of δ-separated δ-tubes in $\mathbb{R}^n$ that satisfy the polynomial Wolff axioms. If $\mathcal{K}_{k,n}(p_0, q_0, e_1, e_2)$ holds, then for every $\delta \leq \rho \leq 1$ and every cap τ_ρ of radius ρ,*

$$\Big\| \sum_{y_i\in\tau_\rho} \chi_{T_i} \Big\|_p \lessapprox \delta^{-(n-1)+((n-1)(p_0-e_2)+e_1)/p}\rho^{(n-1)-((n-1)(p_0-e_2)+\frac{k-1}{k}p_0q_0)/p}$$

provided

$$p \geq \max\Big(p_0,\ 1 + \frac{(n-1)(p_0-e_2) + \frac{k-1}{k}p_0q_0}{n-k+1}\Big).$$

Notice that the case $\rho \sim \delta$ is trivial and that the case $\rho = 1$ gives Proposition 4.1.

To prove this proposition we will follow [2] and break τ_ρ into a covering of smaller caps, $\{\tau_{\rho/K}\}$, each of radius $\leq \rho/K$ (where K is a sufficiently large constant chosen later). A point x will be called broad if for every $(k-1)$-plane, Π,

$$|\{T_i \in \mathbb{T}\ :\ x \in T_i,\ \angle(y_i, \Pi) \geq \rho K^{-1}\}| \geq \frac{1}{2}|\{T_i \in \mathbb{T}\ :\ x \in T_i\}|.$$

A point will be called narrow if it is not broad. Notice that for a narrow point, there are $\lesssim K^{k-2}$ caps that account for $\geq 1/2$ of the tubes. That is, there is a set $C(x) \subset \{\tau_{\rho/K}\}$ with $|C(x)| \lesssim K^{k-2}$ such that

$$\sum_{\tau_{\rho/K}\in C(x)} |\{T_i\ :\ x \in T_i,\ y_i \in \tau_{\rho/K}\}| \geq \frac{1}{2}|\{T\ :\ x \in T\}|.$$

The inductive hypothesis directly covers the narrow points.

Proposition 4.3.

$$\int_{\text{Narrow}} \Big(\sum_{y_i\in\tau_\rho} \chi_{T_i}\Big)^p \lessapprox \delta^{-(n-1)+((n-1)(p_0-e_2)+e_1)/p}\rho^{(n-1)-((n-1)(p_0-e_2)+\frac{k-1}{k}p_0q_0)/p}$$

for

$$p > 1 + \frac{(n-1)(p_0-e_2) + \frac{k-1}{k}p_0q_0}{n-k+1}.$$

Proof. If $x \in$ Narrow, then there exists a collection of caps $C(x) \subset \{\tau_{\rho/K}\}$ with $|C(x)| \lesssim K^{k-2}$ such that

$$\sum_{y_i\in\tau_\rho} \chi_{T_i}(x) \leq 2 \sum_{\tau_{\rho/K}\in C(x)} \sum_{y_i\in\tau_{\rho/K}} \chi_{T_i}(x). \tag{18}$$

By Hölder, (18) implies that

$$\sum_{y_i\in\tau_\rho} \chi_{T_i}(x) \lesssim 2K^{(k-2)(p-1)/p}\Big(\sum_{\tau_{\rho/K}\in C(x)} \Big(\sum_{y_i\in\tau_{\rho/K}} \chi_{T_i}(x)\Big)^p\Big)^{1/p}.$$

Thus,

$$\int_{\text{Narrow}} \Big(\sum_{y_i\in\tau_\rho} \chi_{T_i}\Big)^p \lesssim 2^pK^{(k-2)(p-1)} \sum_{\{\tau_{\rho/K}\}} \int_{B(0,2)} \Big(\sum_{y_i\in\tau_{\rho/K}} \chi_{T_i}(x)\Big)^p. \tag{19}$$

The RHS of (19) is directly controlled by the inductive hypothesis, and thus we get

$$\int_{\mathrm{Narrow}} \Big(\sum_{y_i\in\tau_\rho} \chi_{T_i}\Big)^p \lesssim 2^p K^{(k-2)(p-1)} K^{n-1} \delta^{-(n-1)p+b_1} (\rho/K)^{(n-1)p-((n-1)(p_0-e_2)+\frac{k-1}{k}p_0q_0)}$$
$$\lesssim 2^p K^{(p-1)(k-n-1)+(n-1)(p_0-e_2)+\frac{k-1}{k}p_0q_0} \delta^{-(n-1)p+b_1} \rho^{(n-1)p-((n-1)(p_0-e_2)+\frac{k-1}{k}p_0q_0)}.$$

Provided that

$$(p-1)(k-n-1)+(n-1)(p_0-e_2)+\frac{k-1}{k}p_0q_0 < 0,$$

or equivalently,

$$p > 1 + \frac{(n-1)(p_0-e_2)+\frac{k-1}{k}p_0q_0}{n-k+1}$$

we can close the inductive step by making $K = K(p)$ sufficiently large. $\square$

Proposition 4.4.

$$\int_{\mathrm{Broad}} \Big(\sum_{y_i\in\tau_\rho} \chi_{T_i}\Big)^p \lessapprox \delta^{-(n-1)+((n-1)(p_0-e_2)+e_1)/p} \rho^{(n-1)-((n-1)(p_0-e_2)+\frac{k-1}{k}p_0q_0)/p} \tag{20}$$

provided that

$$p \geq p_0.$$

Proof. Let x be a broad point. By definition of a broad point, most k-tuples of tubes going through x fail to lie near a $(k-1)$-plane. That is to say for most k-tuples of tubes T_1, T_2, …, T_k going through x, we have

$$|v_1 \wedge \cdots \wedge v_k| \geq |y_1 \wedge \cdots \wedge y_k| \geq C(K)\rho^{k-1}.$$

This gives that

$$\Big|\sum_{y_i\in\tau_\rho} \chi_{T_i}\Big|^k \lesssim C(K)^{-1}\rho^{-q_0(k-1)} \sum_{T_1,\ldots,T_k\in\mathbb{T}} \chi_{T_1}\cdots\chi_{T_k}|v_1\wedge\cdots\wedge v_k|^{q_0}$$

and thus

$$\Big|\sum_{y_i\in\tau_\rho} \chi_{T_i}\Big|^{p_0} \lesssim C(K)^{-p_0/k}\rho^{-\frac{k-1}{k}p_0q_0} \Big(\sum_{y_1,\ldots,y_k\in\tau_\rho} \chi_{T_1}\cdots\chi_{T_k}|v_1\wedge\cdots\wedge v_k|^{q_0}\Big)^{p_0/k}. \tag{21}$$

At this point let $\mathbb{T}' = \{T_i \in \mathbb{T} \,:\, y_i \in \tau_\rho\}$. Notice that by the δ-separation hypothesis, $|\mathbb{T}'| \lesssim (\rho/\delta)^{n-1}$. Using the assumption that $\mathcal{K}_{k,n}(p_0,q_0,e_1,e_2)$ holds and letting $\mathbb{T}_1 = \mathbb{T}_2 = \cdots = \mathbb{T}_k = \mathbb{T}'$, we can integrate (21) to get

$$\begin{aligned}\int_{\mathrm{Broad}} \Big(\sum_{y_i\in\tau_\rho} \chi_{T_i}\Big)^{p_0} &\lesssim C(K)^{-p_0/k}\rho^{-\frac{k-1}{k}p_0q_0} \int \Big(\sum_{y_1,\ldots,y_k\in\tau_\rho} \chi_{T_1}\cdots\chi_{T_k}|v_1\wedge\cdots\wedge v_k|^{q_0}\Big)^{p_0/k} \\ &\lessapprox C(K)^{-p_0/k}\rho^{-\frac{k-1}{k}p_0q_0}\delta^{e_1}(\rho/\delta)^{(n-1)e_2} \\ &= C(K)^{-p_0/k}\rho^{(n-1)e_2-\frac{k-1}{k}p_0q_0}\delta^{e_1-(n-1)e_2}.\end{aligned} \tag{22}$$

This gives (20) for the case $p = p_0$. To obtain the full proposition notice that

$$\Big\|\sum_{y_i\in\tau_\rho} \chi_{T_i}\Big\|_\infty \lesssim (\rho/\delta)^{n-1} \tag{23}$$

by the δ-separation hypothesis. The full result then comes from L^p-interpolation between (22) and (23):

$$\int_{\text{Broad}} \Big(\sum_{y_i \in \tau_\rho} \chi_{T_i} \Big)^p \lesssim (\rho^{n-1}\delta^{-(n-1)})^{p-p_0} \int_{\text{Broad}} \Big(\sum_{y_i \in \tau_\rho} \chi_{T_i} \Big)^{p_0}$$
$$\lessapprox C(K)^{-p_0/k} \delta^{-(n-1)p+(n-1)(p_0-e_2)+e_1} \rho^{(n-1)p-(n-1)(p_0-e_2)-\frac{k-1}{k}p_0 q_0}.$$

Realizing that $K = K(p)$, and thus $C(K)$, depends only on p and not the collection of tubes completes the proof. $\square$

Finally, note that the inequality in Proposition 4.3 can be lessened from a strict inequality to a weak one by letting p approach its lower bound and absorbing the loss into the $\delta^{-\epsilon}$ term. This observation combined with Propositions 4.3 and 4.4 proves Proposition 4.2 and thus Theorem 4.1.

As a result of Theorem 4.1, we obtain various linear estimates for collections of bent tubes.

Theorem 4.5. *Let $\mathbb{T}$ be a collection of δ-separated tubes in $\mathbb{R}^n$ for $n \geq 3$, then*

$$\Big\| \sum_{T \in \mathbb{T}} \chi_T \Big\|_p \lessapprox \delta^{-(n-1)+n/p}$$

for

$$p \geq \begin{cases} \frac{n+2}{n} & n \text{ even} \\ \frac{n+1}{n-1} & n \text{ odd} \end{cases}.$$

Proof. Apply Theorem 4.1 letting $k = n/2 + 1$ if n is even and $k = (n+1)/2$ if n is odd, using Theorem 4 to assume that $\mathcal{K}_{k,n}\big(\frac{k}{k-1},\ 1,\ n\ ,\frac{k}{k-1}\big)$ holds. $\square$

Remark 4.1. This recovers Wolff's classical bound with an error term of $\delta^{-\epsilon}$ when n is even.

It is at this point that we remark on the utility of the multilinear bounds obtained using the polynomial Wolff axioms. As it works out, the original endpoint cases of the Bennett-Carbery-Tao multilinear estimates given by Theorem 4 produce better linear estimates than those given by Theorem 2.6. This is due to the fact that Proposition 3.3 gives an improvement to the input bound for large sets of tubes at the expense of a weaker bound for small sets of tubes. This in effect undermines the inductive element used to prove the bound on narrow points resulting in worse bounds.

5 Mixing Multilinear Bounds

In the previous section we showed that the method introduced by Guth and Bourgain in [2] fails to make good use of the improved multilinear bounds found by Guth and Zahl. However, it remains that the improved multilinear bounds are at the heart of Guth and Zahl's improved maximal function estimate in $\mathbb{R}^4$. A way to circumvent the problems in the previous section would be to take an approach similar to that used in [5].

The general approach is to strengthen the multilinear bounds further (via an approach applicable to any collection of multilinear estimates) by deriving in a sense "anti"-multilinear bounds, i.e. bounds that exploit sets of tubes that are k-linear but fail to be $(k+1)$-linear.

Assume we want to improve a known k-linear bound. To gain extra mileage we assume that our collections of tubes are k-linear but strongly fail to be $(k+1)$-linear—otherwise we would simply apply the $(k+1)$-linear bound. Adding this assumption we can improve the k-linear bound. In the end, we find that a collection of tubes that is k-linear will either strongly fail to be $(k+1)$-linear or nearly be $(k+1)$-linear, resulting in a new k-linear bound that lies somewhere between the original k-linear bound and the $(k+1)$-linear bound. In [5], Guth and Zahl use this technique to improve the bilinear bound. By combining this with the fact that collections of 1-linear tubes are easily handled by induction on scales (allowing us to assume most pairs of tubes point in linearly independent directions), they simply apply the improved bilinear bound and conclude their result. We sketch an outline to this technique.

We begin by introducing a couple definitions to help formalize this notion.

Definition 5.1 (Shadings)**.** A shading of a tube T is any subset of T. We will denote a collection of tubes, each having an associated shading, by $(\mathbb{T}, Y)$ where for every $T \in \mathbb{T}$, $Y(T)$ is the shading associated to T.

Definition 5.2 (Robust Transversality)**.** A collection of tubes with shadings $(\mathbb{T}, Y)$ is said to be (s, k)-robustly transverse if for every $(k-1)$-plane Π and every x,

$$|\{T_i \in \mathbb{T} \,:\, x \in Y(T_i),\ \ \angle(v_i(x), \Pi) \geq s\}| \gtrsim |\{T \in \mathbb{T} \,:\, x \in Y(T)\}|.$$

Definition 5.3 (Weak Transversality)**.** A collection of tubes with shadings $(\mathbb{T}, Y)$ is said to be (s, k)-weakly transverse if for every $(k-1)$-plane Π and every x,

$$|\{T_i \in \mathbb{T} \,:\, x \in Y(T_i),\ \ \angle(v_i(x), \Pi) < s\}| \gtrsim |\{T \in \mathbb{T} \,:\, x \in Y(T)\}|.$$

To improve an existing k-linear bound we will then consider the dichotomy of when the collection of tubes is $(\theta, k+1)$-robustly transverse and when it $(\theta, k+1)$-weakly transverse. By choosing an appropriate θ these two bounds will meet, and for any general collection of tubes the quantity of interest will either be dominated by the contribution of robustly transverse points or weakly transverse points.

Theorem 5.4. *Suppose that $\mathcal{K}_{k+1,n}(p_0, q_0, e_1, e_2)$ holds. Let $(\mathbb{T}, Y)$ be a collection of tubes with shadings that are (s, k)-robustly transverse and $(\theta, k+1)$-robustly transverse. If for every $T \in \mathbb{T}$ we have that $|Y(T)| \sim \lambda |T|$, then*

$$\Big|\bigcup_{T\in\mathbb{T}} Y(T)\Big| \gtrapprox \Big(s^{p_0 q_0 \frac{k-1}{k+1}} \theta^{\frac{p_0 q_0}{k+1}} \lambda^{p_0} \delta^{(n-1)e_2 - e_1} (\delta^{n-1}|\mathbb{T}|)^{p_0 - e_2}\Big)^{\frac{1}{p_0-1}}.$$

Proof. After a refinement of the shadings we may assume that $\sum \chi_T \sim \mu \chi_B$. By the assumptions on the robust transversality we have that for every $x \in B$,

$$\sum_{T_1,\ldots,T_{k+1}\in\mathbb{T}} \chi_{T_1}(x)\ldots\chi_{T_{k+1}}(x)|v_1(x)\wedge\cdots\wedge v_{k+1}(x)|^{q_0} \gtrsim \mu^{k+1} s^{q_0(k-1)} \theta^{q_0}. \tag{24}$$

To see this, there are $\gtrsim \mu$ choices for T_1. By the fact that the collection is (s,k)-robustly transverse, we have $\gtrsim \mu$ options for T_2 such that $\angle(v_2(x), v_1(x)) \geq s$, $\gtrsim \mu$ options for T_3 such that $\angle(v_3(x), \mathrm{span}(v_1(x), v_2(x))) \geq s$, and so on. Finally, since the collection of tubes is $(\theta, k+1)$-robustly transverse, there are $\gtrsim \mu$ choices for T_{k+1} such that $\angle(v_{k+1}(x), \mathrm{span}(v_1(x), \ldots, v_k(x))) \geq \theta$. Thus there are $\gtrsim \mu^{k+1}$ choices for $T_1, \ldots, T_{k+1}$ all which obeying $|v_1 \wedge \cdots \wedge v_{k+1}| \gtrsim s^{k-1}\theta$.

Integrating (24) gives

$$\int_B \Big(\sum_{T_1,\ldots,T_{k+1}\in\mathbb{T}} \chi_{T_1}(x)\ldots\chi_{T_{k+1}}(x)|v_1(x)\wedge\cdots\wedge v_{k+1}(x)|^{q_0}\Big)^{p_0/(k+1)} \gtrsim |B|\mu^{p_0} s^{p_0 q_0(k-1)/(k+1)} \theta^{q_0 p_0/(k+1)}. \tag{25}$$

However, by the assumption that $\mathcal{K}_{k+1,n}(p_0, q_0, e_1, e_2)$ holds we also have

$$\int_B \Big(\sum_{T_1,\ldots,T_{k+1}\in\mathbb{T}} \chi_{T_1}(x)\ldots\chi_{T_{k+1}}(x)|v_1(x)\wedge\cdots\wedge v_{k+1}(x)|^{q_0}\Big)^{p_0/(k+1)} \lessapprox \delta^{e_1}|\mathbb{T}|^{e_2}. \tag{26}$$

Combining (25) and (26) gives

$$|B|\mu^{p_0} s^{p_0 q_0(k-1)/(k+1)} \theta^{q_0 p_0/(k+1)} \lessapprox \delta^{e_1}|\mathbb{T}|^{e_2}.$$

Using this and the fact that $|B| \approx \lambda\delta^{n-1}|\mathbb{T}|\mu^{-1}$ we have

$$\Big|\bigcup_{T\in\mathbb{T}} Y(T)\Big| \approx |B| \gtrapprox \left(s^{p_0 q_0 \frac{k-1}{k+1}}\theta^{\frac{p_0 q_0}{k+1}}\lambda^{p_0}\delta^{(n-1)e_2-e_1}(\delta^{n-1}|\mathbb{T}|)^{p_0-e_2}\right)^{\frac{1}{p_0-1}}.$$

$\square$

The goal now is to prove the complementary "anti"-$(k+1)$-linear bound: One where we assume that the tubes are $(\theta, k+1)$-weakly transverse. This is where new techniques are required. To demonstrate the strategy we give an example of such a theorem.

Example 5.1. *Assume that $\mathcal{K}_{k,n}(p_0, q_0, e_1, e_2)$ holds. Let $(\mathbb{T}, Y)$ be a collection of tubes with shadings that are (s,k)-robustly transverse and $(\theta, k+1)$-weakly transverse. If for every $T \in \mathbb{T}$ we have that $|Y(T)| \sim \lambda|T|$, then*

$$\Big|\bigcup_{T\in\mathbb{T}} Y(T)\Big| \gtrapprox s^{m_1}\theta^{-m_2}\lambda^{m_3}\delta^{m_4}(\delta^{n-1}|\mathbb{T}|)^{m_5}.$$

Remark 5.1. Here all the exponents $m_1, \dots, m_5$ are assumed to be non-negative. It is of note that in this complementary bound, the exponent on θ is negative. Since $\theta \lesssim 1$, a negative exponent implies a stronger bound. This matches our intuition: Had we dropped the assumption that the collection of tubes be $(\theta, k+1)$-weakly transverse we could simply use the fact that they are (s,k)-robustly transverse and apply the k-linear bound, ignoring the θ term. The fact that the θ term has a negative exponent, and thus strengthens the bound, reflects that assuming the tubes are $(\theta, k+1)$-weakly transverse adds new information.

We finally present a manner by which to combine these two complementary bounds.

Theorem 5.5. *Suppose that for all collections of tubes with shadings $(\mathbb{T}, Y)$ that are (s,k)-robustly transverse and $(\theta, k+1)$-robustly transverse, we have that*

$$\Big|\bigcup_{T\in\mathbb{T}} Y(T)\Big| \gtrapprox s^{m_1}\theta^{m_2}\lambda^{m_3}\delta^{m_4}(\delta^{n-1}|\mathbb{T}|)^{m_5}, \tag{27}$$

and for all collections of tubes with shadings $(\mathbb{T}, Y)$ that are (s,k)-robustly transverse and $(\theta, k+1)$-weakly transverse, we have that

$$\Big|\bigcup_{T\in\mathbb{T}} Y(T)\Big| \gtrapprox s^{m_1'}\theta^{-m_2'}\lambda^{m_3'}\delta^{m_4'}(\delta^{n-1}|\mathbb{T}|)^{m_5'}. \tag{28}$$

Then, for all collections of tubes with shadings $(\mathbb{T}, Y)$ that are (s,k)-robustly transverse, we have that

$$\Big|\bigcup_{T\in\mathbb{T}} Y(T)\Big| \gtrapprox \lambda^{\frac{m_3 m_2' + m_3' m_2}{m_2 + m_2'}}\delta^{\frac{m_4 m_2' + m_4' m_2}{m_2 + m_2'}}\min(s^{m_1}(\delta^{n-1}|\mathbb{T}|^{m_5}), s^{m_1'}(\delta^{n-1}|\mathbb{T}|^{m_5'})).$$

Remark 5.2. Notice that this theorem allows us to gain a new k-linear bound. The assumption that the tubes be (s,k)-robustly transverse for s sufficiently large is in essence saying most k-tuples of tubes point in k linearly independent directions.

Proof. Let

$$\theta_0 = \left(\delta^{m_4'-m_4}\lambda^{m_3'-m_3}\right)^{\frac{1}{m_2+m_2'}}$$

and let X_1 be the set of all point such that there exists a k-plane Π such that

$$|\{T_i \in \mathbb{T} \,:\, x \in Y(T_i),\ \angle(v_i(x), \Pi) < \theta_0\}| \gtrsim |\{T_i \in \mathbb{T} \,:\, x \in Y(T_i)\}|.$$

If $X_2 = \mathbb{R}^n \setminus X_1$ then we know that either

$$\sum_{T\in\mathbb{T}} |Y(T) \cap X_1| \gtrsim \lambda\delta^{n-1}|\mathbb{T}| \tag{29}$$

or

$$\sum_{T\in\mathbb{T}}|Y(T)\cap X_2|\gtrapprox\lambda\delta^{n-1}|\mathbb{T}| \tag{30}$$

holds.

If (29) holds then we can replace each shading with $Y'(T)=Y(T)\cap X_1$ and find a set $\mathbb{T}'\subset\mathbb{T}$ with $|\mathbb{T}'|\sim|\mathbb{T}|$ and $Y'(T)\sim\lambda|T|$ for every $T\in\mathbb{T}'$. By (28) we have that

$$\Big|\bigcup_{T\in\mathbb{T}}Y(T)\Big|\geq\Big|\bigcup_{T\in\mathbb{T}'}Y'(T)\Big|\gtrapprox s^{m_1'}\lambda^{\frac{m_3m_2'+m_3'm_2}{m_2+m_2'}}\delta^{\frac{m_4m_2'+m_4'm_2}{m_2+m_2'}}(\delta^{n-1}|\mathbb{T}|)^{m_5'}.$$

Similarly, if (30) holds then we can replace each shading with $Y'(T)=Y(T)\cap X_2$ and find a set $\mathbb{T}'\subset\mathbb{T}$ with $|\mathbb{T}'|\sim|\mathbb{T}|$ and $Y'(T)\sim\lambda|T|$ for every $T\in\mathbb{T}'$. By (27) we have that

$$\Big|\bigcup_{T\in\mathbb{T}}Y(T)\Big|\geq\Big|\bigcup_{T\in\mathbb{T}'}Y'(T)\Big|\gtrapprox s^{m_1}\lambda^{\frac{m_3m_2'+m_3'm_2}{m_2+m_2'}}\delta^{\frac{m_4m_2'+m_4'm_2}{m_2+m_2'}}(\delta^{n-1}|\mathbb{T}|)^{m_5}.$$

□

To go to a linear bound from these improved multilinear bounds there is always the naive method of proving that we may assume the collections of tubes are (s,k)-robustly transverse for some large s and directly applying the k-linear bound from Theorem 5.5. This is the route of Guth and Zahl in [5]. Since tubes pointing in a single direction can be easily handled by induction on scales, one can assume $(s,2)$-robust transversality, and directly apply the 2-linear bound obtained from the technique above.

6 Future Work

It remains unknown what the optimal multilinear bounds are under the assumption of the polynomial Wolff axioms. As it currently stands, the best techniques are agnostic towards whether the tubes are straight or bent, but it is unknown whether there will exist a disparity between optimal bounds for straight versus bent tubes. However, if methods for proving optimal bounds are of a similar flavor to existing ones, it is very possible that multilinear bounds produced under the polynomial Wolff axioms extend to bent tubes.

If this is the case, then the straightness constraint on the tubes may lie squarely in going from multilinear bounds to linear bounds. Another possibility is that the straightness hypothesis could only be necessary so that, when combined with δ-separation, the polynomial Wolff axioms hold: the implication of the polynomial Wolff axioms from straightness and δ-separation was proven by Katz and Rogers in [7].

Results regarding complementary multilinear bounds, i.e. multilinear bounds that utilize information about the tubes lying close to a k-plane, could also play an important role. Doing this would open the doors to better multilinear bounds which could serve to gain improvements on estimates for both straight and bent tubes.

7 Acknowledgments

I am extremely grateful to my mentor Mr. Robert Burklund for helpful discussions on the problem and assistance throughout the research, both in familiarizing me with the problem and providing me with intuition. I would also like to thank Prof. Larry Guth for proposing the problem and the MIT PRIMES-USA program for the opportunity to conduct this research.

References

[1] Jonathan Bennett, Anthony Carbery, and Terence Tao. On the multilinear restriction and Kakeya conjectures. *Acta mathematica*, 196(2):261–302, 2006.

[2] Jean Bourgain and Larry Guth. Bounds on oscillatory integral operators based on multilinear estimates. *Geometric and Functional Analysis*, 21(6):1239–1295, 2011.

[3] Zeev Dvir. On the size of Kakeya sets in finite fields. *Journal of the American Mathematical Society*, 22(4):1093–1097, 2009.

[4] Larry Guth and Nets Hawk Katz. On the Erdos distinct distances problem in the plane. *Annals of mathematics*, 181(1):155–190, 2015.

[5] Larry Guth and Joshua Zahl. Polynomial Wolff axioms and Kakeya-type estimates in $\mathbb{R}^4$. *arXiv preprint arXiv:1701.07045*, 2017.

[6] Nets Katz and Terence Tao. Recent progress on the Kakeya conjecture. *arXiv preprint math/0010069*, 2000.

[7] Nets Hawk Katz and Keith M Rogers. On the polynomial Wolff axioms. *arXiv preprint arXiv:1802.09094*, 2018.

[8] Thomas H Wolff. An improved bound for Kakeya type maximal functions. *Revista Matemática Iberoamericana*, 11(3):651–674, 1995.

[9] Richard Wongkew. Volumes of tubular neighbourhoods of real algebraic varieties. *Pacific Journal of Mathematics*, 159(1):177–184, 1993.