Smoothness of the logarithmic Hodge moduli space

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1 Preliminaries

We quickly recall the notion of *hypercohomology*. For a space X and a sheaf $\mathcal{F}$ on X, we define $H^k(X, \mathcal{F})$ by taking an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ and setting

$$H^*(X, \mathcal{F}) = H^*(\Gamma(X, \mathcal{I}^\bullet)).$$

More generally, notice that a resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ is simply a quasi-isomorphism (i.e. a chain map inducing an isomorphism on cohomology) between the complex of sheaves $\mathcal{F}$ supported in degree zero and $\mathcal{I}^\bullet$. Thus, to take the cohomology of a complex $\mathcal{F}^\bullet$ of sheaves on X not-necessarily supported in degree zero, we should take a quasi-isomorphism $\mathcal{F}^\bullet \xrightarrow{\simeq} \mathcal{I}^\bullet$, where each $\mathcal{I}^i$ is injective, and set

$$\mathbb{H}^*(X, \mathcal{F}^\bullet) = H^*(\Gamma(X, \mathcal{I}^\bullet)).$$

This is just (up to taking cohomology) the derived functor of global sections, viewed as a functor $\mathrm{Ch}(\mathrm{Shv}_{\mathcal{A}\mathrm{b}}(X)) \to \mathrm{Ch}(\mathcal{A}\mathrm{b})$, i.e. $\mathbb{H}^*(X, \mathcal{F}^\bullet) = H^*(R\Gamma(X, \mathcal{F}^\bullet))$.

2 Introduction

Let X be a smooth complex projective variety. By applying GAGA and using (p, q)-decomposition of complex differential forms, one may show that the hypercohomology of the complex of holomorphic or algebraic differential forms computes the usual de Rham cohomology, i.e.

$$\mathbb{H}^k(X, (\Omega_X^\bullet, d)) \cong H^k_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbb{C}).$$

Moreover, by formal manipulations, we have that

$$\begin{aligned}
\mathbb{H}^k(X, (\Omega_X^\bullet, 0)) &= \mathbb{H}^k(X, \bigoplus_i \Omega_X^i[-i]) \\
&\cong \bigoplus_i \mathbb{H}^{k-i}(X, \Omega_X^i) \\
&\cong \bigoplus_{p+q=k} H^q(X, \Omega_X^p).
\end{aligned}$$

Using these, one way to state the classical Hodge decomposition is that

$$\mathbb{H}^k(X, (\Omega_X^\bullet, d)) \cong \mathbb{H}^k(X, (\Omega_X^\bullet, 0)). \tag{2.0.1}$$

Written this way, one naturally asks whether there exists some more fundamental relation between the objects $(\Omega_X^\bullet, d)$ and $(\Omega_X^\bullet, 0)$ from which the Hodge decomposition follows by passing to cohomology. The non-abelian Hodge theorem provides such a result by categorifying the classical Hodge decomposition.

To arrive at such a result, one should figure out how to view $(\Omega_X^\bullet, d)$ and $(\Omega_X^\bullet, 0)$ geometrically.

Let $\mathcal{F}$ be a vector bundle on X and let ∇ be a flat connection on $\mathcal{F}$, i.e. a $\mathbb{C}$-linear morphism $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^1$ satisfying the Leibniz rule. Then one may naturally, by flatness of ∇, associate to $(\mathcal{F}, \nabla)$ the complex

$$\mathscr{F} \xrightarrow{\nabla} \mathcal{F} \otimes \Omega_X^1 \xrightarrow{\nabla} \mathcal{F} \otimes \Omega_X^2 \xrightarrow{\nabla} \cdots.$$

Doing this, we may view $(\Omega_X^\bullet, d)$ as arising from the trivial line bundle $\mathcal{O}_X$ with flat connection d.

This suggests that $(\Omega_X^\bullet, 0)$ may also be viewed as a complex arising from the trivial line bundle on X with some additional structure. The correct structure to consider ends up being the notion of a *Higgs field*.

Definition 2.1. Let $\mathcal{F}$ be a vector bundle on X. A *Higgs field* on $\mathcal{F}$ is a section $\varphi \in \Gamma(X, \mathrm{End}(\mathcal{F}) \otimes \Omega_X^1)$ which commutes with itself, i.e. $\varphi \wedge \varphi = 0$. The pair $(\mathcal{F}, \varphi)$ is referred to as a *Higgs bundle*.

Given a Higgs bundle $(\mathcal{F}, \varphi)$ on X, we may view φ as a morphism $\varphi : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^1$ and consider the complex

$$\mathcal{F} \xrightarrow{\varphi} \mathcal{F} \otimes \Omega_X^1 \xrightarrow{\varphi} \mathcal{F} \otimes \Omega_X^2 \xrightarrow{\varphi} \cdots \tag{2.0.2}$$

where we extend φ to morphisms $\mathcal{F} \otimes \Omega_X^n \to \mathcal{F} \otimes \Omega_X^{n+1}$ by acting on the $\mathcal{F}$ component. The fact that $\varphi \wedge \varphi = 0$ ensures that (2.0.2) is a complex. Note that $(\mathcal{O}_X, 0)$ is then a Higgs bundle with associated complex $(\Omega_X^\bullet, 0)$.

One may then hope that the correct form of the NAHT is an equivalence between the two categories

$$\left\{ \begin{array}{c} \text{vector bundles with} \\ \text{flat connection over } X \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Higgs bundles} \\ \text{over } X \end{array} \right\}$$

which preserves the cohomology of the naturally associated complexes, and for which $(\mathcal{O}_X, d)$ corresponds to $(\mathcal{O}_X, 0)$. Up to some stability conditions, this is true.

Theorem 2.2 (Non-Abelian Hodge Theorem)**.** *Let X be a smooth complex projective variety. Up to stability conditions, there is a cohomology preserving equivalence between*

(i) flat bundles $(\mathcal{F}, \nabla)$ over X

(ii) Higgs bundles $(\mathcal{F}, \varphi)$ over X.

3 Where is the AG?

It turns out that the classical NAHT is a fundamentally smooth result—it is proven using analysis and then given to us algebraically via GAGA. However, a weakened version does lend itself to algebraic approaches over more general base rings. It turns out that the two objects featured in the NAHT fit into natural moduli spaces, and the NAHT provides an isomorphism between these spaces. As such, they have isomorphic cohomologies. One may thus seek a "cohomological NAHT" over more general base rings, i.e. showing that the two natural moduli spaces occuring in the NAHT have isomorphic cohomologies.

One approach is to consider a family of geometric objects living over X, smoothly varying in some parameter(s), which interpolate between flat bundles and Higgs bundles. This is achieved via t-connections, originally suggested by Deligne.

Definition 3.1. Let X/B be a smooth scheme. Given a vector bundle $\mathcal{F}$ on X, a t-connection (for $t \in \Gamma(B, \mathcal{O}_B)$) on $\mathcal{F}$ is an $\mathcal{O}_B$-linear morphism

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_{X/B}$$

of sheaves satisfying the t-twisted Leibniz rule

$$\nabla(fs) = ts \otimes df + f\nabla s$$

for all $f \in \mathcal{O}_X$ and $s \in \mathcal{F}$.

Given a t-connection ∇ on $\mathcal{F}$, we may extend it to morphisms $\mathcal{F} \otimes \Omega^n_{X/B} \to \mathcal{F} \otimes \Omega^{n+1}_{X/B}$ by enforcing the t-twisted Leibniz rule

$$\nabla(s \otimes w) = ts \otimes dw + (\nabla s) \wedge w$$

locally. We call a t-connection ∇ flat if $\nabla \circ \nabla = 0$.

With this definition, a Higgs field is simply a flat 0-connection and a flat connection is a flat 1-connection. This gives a family of geometric objects over $\mathbb{A}^1_B$ interpolating between Higgs bundles and flat bundles. If we then consider the moduli space MHodge_X of all bundles with t-connection on X, we get a morphism

$$\mathrm{MHodge}_X \to \mathbb{A}^1_B.$$

One approach to a cohomological NAHT is to prove that this morphism is smooth, as then we can then relate the cohomologies of various fibres via "specialization maps" (which generically only exist in the proper setting, but are constructed here in [DC22]).

4 The logarithmic setting

It turns out that it is best to consider connections with potential poles along a fixed divisor. We consider the following setup:

- B is a Noetherian scheme
- C/B is a smooth proper morphism of schemes with geometrically integral fibers of dimension 1
- $D \to C$ is a relative Cartier divisor such that every geometric B-fiber of D is non-empty (this is important!) and reduced
- n and d are coprime integers

We will write C/B as C_B, and given a B-scheme S, we will write C_S for the S-scheme $C \times_B S \to S$.

Definition 4.1. Denote by $\mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_B} \to \mathbb{A}^1_B$ the moduli stack of (slope) semistable rank n and degree d t-connections. As a semifunctor, this assigns to an $\mathbb{A}^1_B$-scheme S the groupoid of pairs $(\mathcal{F}, \nabla)$ where:

(i) $\mathcal{F}$ is a vector bundle of rank n on C_S such that the restriction to each geometric fiber of $C_S \to S$ has degree d

(ii) $\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{C_S}} \omega_{C_S/S}(D_S)$ is a logarithmic t_S-connection with at most simple poles allowed on the pullback D_S of D

(iii) The restriction of the pair $(\mathcal{F}, \nabla)$ to each geometric fiber C_s of the morphism $C_S \to S$ is a semistable t_s-connection.

We may also consider the moduli space

$$\mathrm{MHodge}^{\mathrm{ss}}_{C_B} \to \mathbb{A}^1_B \tag{4.0.1}$$

of rank n and degree d semistable logarithmic t-connections over B constructed via Geometric Invariant Theory. Our goal is to prove that (4.0.1) is smooth. To do this, we wish to reduce to proving the smoothness of

$$\mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_B} \to \mathbb{A}^1_B.$$

To achieve this, we show that

$$\mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_B} \to \mathrm{MHodge}^{\mathrm{ss}}_{C_B}$$

is a smooth surjection via the following more general proposition.

Proposition 4.2. *The natural map* $\mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_B} \to \mathit{MHodge}^{\mathrm{ss}}_{C_B}$ *is a smooth good moduli space morphism (in the sense of [Alp13]).*

Sketch of Idea. There is a central copy of $\mathbb{G}_m$ in the automorphisms of every point of $\mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_B}$ since multiplication by constants commutes with any logarithmic t-connection. We may thus universally remove these automorphisms by taking a *rigidification* in the sense of [AOV08, Appendix A]. Call this stack $(\mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_B})^{\mathrm{rig}}$. The rigidification morphism $\mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_B} \to (\mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_B})^{\mathrm{rig}}$ is a smooth good moduli space morphism.

Using the coprimality of n and d, which implies semistable t-connections are stable, one may then show that $(\mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_B})^{\mathrm{rig}}$ is in fact an algebraic space. The universal property of rigification then induces a factoring

$$\begin{array}{ccc} \mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_B} & \longrightarrow & (\mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_B})^{\mathrm{rig}} \\ & \searrow \quad \swarrow_{\psi} & \\ & \mathrm{MHodge}^{\mathrm{ss}}_{C_B} & \end{array}$$

and one may show that, using properties of good moduli space morphisms and the fact that $(\mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_B})^{\mathrm{rig}}$ is an algebraic space, that ψ is an isomorphism. $\square$

Since good moduli morphisms are surjective and smoothness may be checked smooth locally, Proposition 4.2 gives the following corollary.

Corollary 4.3. *If the rank n and degree d are coprime,* $\mathit{MHodge}^{\mathrm{ss}}_{C_B} \to \mathbb{A}^1_B$ *is smooth if and only if* $\mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_B} \to \mathbb{A}^1_B$ *is smooth.*

We then have the following theorem.

Theorem 4.4. *For general n and d (not necessarily coprime), the morphism $\mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_B} \to \mathbb{A}^1_B$ is smooth.*

Sketch of Idea. We show smoothness by proving the existence of lifting for square-zero thickenings of local Artin algebras.

To facilitate this, [dCHZ24] lays out a deformation theory of t-connections. In particular, for every morphism $x_A : \mathrm{Spec}(A) \to \mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_B}$ they construct an obstruction module $\mathcal{Q}_{x_A}$ in which the obstruction to a lifting

$$\begin{array}{ccc} \mathrm{Spec}\, A & \longrightarrow & \mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_B} \\ \big\downarrow & \nearrow & \big\downarrow \\ \mathrm{Spec}\, \widetilde{A} & \longrightarrow & \mathbb{A}^1_B \end{array}$$

along a square-zero thickening $\mathrm{Spec}\, A \to \mathrm{Spec}\, \widetilde{A}$ exists. Thus it suffices to show each obstruction module $\mathcal{Q}_{x_A}$ is zero, where A is a local Artinian algebra.

By a series of reductions, we may reduce to the case where $B = \mathrm{Spec}\, k$, k algebraically closed, and $x_k : \mathrm{Spec}\, k \to \mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_k}$ is a geometric point.

Using semistable reduction arguments, this geometric point may be extended to a family $x_{\mathbb{A}^1_k} : \mathbb{A}^1_k \to \mathcal{M}\mathrm{Hodge}^{\mathrm{ss}}_{C_k}$ restricting to x_k at $1 \in \mathbb{A}^1_k$ and compatible with the rescaling action on t-connections. If x_p denotes the restriction of $x_{\mathbb{A}^1_k}$ to $p \in \mathbb{A}^1_k$, then by transporting the obstruction module along $x_{\mathbb{A}^1_k}$, it suffices to show that $\mathcal{Q}_{x_0} = 0$.

Since x_0 represents a Higgs bundle, we are thus reduced to showing the vanishing of obstruction modules at points representing Higgs bundles. This requires an explicit look at the obstruction module and in particular makes explicit use of the fact that the divisor $D \to C \to B$ has non-empty fibers over every geometric B-point. $\square$

References

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