

Thom twists, six functor formalisms and semi-additivity

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I Motivation: Six (three) functor formalisms

Let \mathcal{C} be a category of geometric objects which has finite limits, and $E \subseteq \mathcal{C}$ a wide subcategory which is preserved under pullbacks.

Definition 1.1. A three functor formalism on \mathcal{C} is a lax monoidal functor

$$\mathcal{D} : \text{Span}(\mathcal{C}, E) \longrightarrow \text{Cat}$$

where Cat is given the Cartesian monoidal structure and $\text{Span}(\mathcal{C})$ is given the induced monoidal structure coming from the Cartesian monoidal structure on \mathcal{C} and the fact that $\text{Span}(\mathcal{C})$ preserves products.

For this talk, let us table the lax monoidality in Definition 1.1. For those interested, lax monoidality ensures that, in what follows,

- (i) each $\mathcal{D}(x)$ is symmetric monoidal
- (ii) the pullback functors f^* are symmetric monoidal
- (iii) the exceptional pushforwards satisfy a projection formula.

However, one may show that the symmetric monoidal category $\text{Span}(\mathcal{C})^\otimes$ is *itself* a span category. Thus understanding how to construct ordinary functors $\text{Span}(\mathcal{C}) \rightarrow \mathcal{D}$ for arbitrary \mathcal{C} will already inform how to construct lax monoidal functors $\text{Span}(\mathcal{C}) \rightarrow \mathcal{D}$.

That said, suppose we have an ordinary functor

$$\mathcal{D} : \text{Span}(\mathcal{C}) \longrightarrow \text{Cat}.$$

What sort of data does such a functor encode? For every $f : x \rightarrow y \in \mathcal{C}$, we obtain a map

$$f^* = \mathcal{D} \left(\begin{array}{ccc} & x & \\ f \swarrow & & \searrow \\ y & & x \end{array} \right) : \mathcal{D}(y) \longrightarrow \mathcal{D}(x)$$

which we refer to as *pullback*. Additionally, whenever $f : x \rightarrow y \in E \subseteq \mathcal{C}$, we obtain a map

$$f_! = \mathcal{D} \left(\begin{array}{ccc} & x & \\ \swarrow & & \searrow f \\ y & & x \end{array} \right) : \mathcal{D}(x) \longrightarrow \mathcal{D}(y)$$

which we refer to as *exceptional pushforward*. For a general span, we may factor it as

$$\begin{array}{c}
 z \\
 \swarrow f \quad \searrow g \\
 x \quad \quad y
 \end{array}
 =
 \begin{array}{c}
 z \\
 \swarrow f \\
 x \\
 \parallel \\
 z \\
 \parallel \\
 z \\
 \searrow g \\
 y
 \end{array}$$

to see that

$$\mathcal{D} \left(\begin{array}{c} z \\ \swarrow f \quad \searrow g \\ x \quad \quad y \end{array} \right) = g!f^* : \mathcal{D}(x) \longrightarrow \mathcal{D}(y).$$

The pullback and exceptional pushforward cover all the data contained in 1-simplices in \mathcal{D} .

The composition in $\text{Span}(\mathcal{C}, E)$ encodes a *base change theorem* for $(-)_!$ and $(-)^*$. More precisely, suppose that we have a pullback square

$$\begin{array}{ccc}
 x & \xrightarrow{\bar{g}} & y' \\
 \bar{f} \downarrow & \lrcorner & f \downarrow \\
 y & \xrightarrow{g} & z
 \end{array}$$

with $f, \bar{f} \in E$. This supplies a 2-cell

$$\begin{array}{ccccc}
 & & x & & \\
 & & \swarrow \bar{f} & \searrow \bar{g} & \\
 & y & & & y' \\
 & \parallel & \searrow g & \swarrow f & \parallel \\
 y & & z & & y'
 \end{array}$$

of $\text{Span}(\mathcal{C}, E)$. Since \mathcal{D} must respect composition, \mathcal{D} provides an equivalence

$$\bar{g}! \bar{f}^* \simeq f^* g!.$$

The higher simplices ensure that these equivalence are compatible with pasting of Cartesian squares in \mathcal{C} .

It turns out this *fully* describes what a functor out of $\text{Span}(\mathcal{C}, E)$ looks like, as the following theorem, which we will hopefully revisit later, makes precise.

Theorem 1.2 (Liu–Zheng [LZ12]). *Span*(\mathcal{C}, E) is the category freely generated by a copy of \mathcal{C}^{op} and E modulo the relations generated by pullbacks in \mathcal{C} . In particular, a functor $\text{Span}(\mathcal{C}, E) \rightarrow \mathcal{D}$ is equivalent to the data of

- (i) a contravariant functor $F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$
- (ii) a covariant functor $G : E \rightarrow \mathcal{D}$
- (iii) the data of a coherent base change theorem between F and G

2 Constructing six functor formalisms

So how does one construct a six functor formalism in practice? Typically, one starts with a pullback functor

$$\mathcal{D} : \mathcal{C}^{\text{op}} \longrightarrow \text{Cat}$$

and is tasked with defining the exceptional pushforward for a class of morphisms E .

For this, one distinguishes two wide subcategories $I, P \subseteq E$ which one should think of as

$$I = \text{“open immersions”}$$

$$P = \text{“proper maps”}.$$

Then, one proves a compactification result which says that every morphism $f \in E$ may be factored as $p \circ i$ for some $i \in I$ and $p \in P$. The exceptional pushforward is then prescribed as follows:

- (i) for $i \in I$, we take $i_! = i_{\natural} \dashv i^*$ to be the left adjoint of pullback
- (ii) for $p \in P$, we take $p_! = p_* \vdash p^*$ to be the right adjoint of pullback
- (iii) for $f \in E$, we factor $f = p \circ i$ and set $f_! = p_! \circ i_!$

This has (at least) two immediate red flags, namely:

- for $f \in I \cap P$, we need a canonical identification $\text{Nm}_f : f_{\natural} \longrightarrow f_*$ —this is semi-additivity!
- we’ve assumed nothing about the factorizations of $f \in E$, so we need to coherently identify $p_* \circ i_{\natural}$ for *every* possible factorization $f = p \circ i$.

At first glance, the second issue seems almost insurmountable in an ∞ -categorical context. However, it turns out that both issues may be resolved with very minor hypotheses! This is the content [LZ12].

We make the following working assumptions, and we state what they correspond to in terms of the category theory of the parametrized category $\mathcal{D} : \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$:

(H1) I and P are closed under base change

- The morphisms in I and P generate a local class of morphisms in $\text{PSh}(\mathcal{C})$, hence determine subuniverses Ω_I and Ω_P of the universe Ω of $\text{PSh}(\mathcal{C})$ -groupoids in the sense of [MW21] over which we may consider colimits and limits, respectively.

(H2) For every pullback square

$$\begin{array}{ccc} x & \xrightarrow{\bar{g}} & y' \\ \bar{i} \downarrow & \lrcorner & i \downarrow \\ y & \xrightarrow{g} & z \end{array}$$

in \mathcal{C} with $i, \bar{i} \in I$, the square

$$\begin{array}{ccc} \mathcal{D}(x) & \xleftarrow{\bar{g}^*} & \mathcal{D}(y') \\ \bar{i}^* \uparrow & \lrcorner & i^* \uparrow \\ \mathcal{D}(y) & \xleftarrow{g^*} & \mathcal{D}(z) \end{array}$$

is vertically left adjointable

- The parametrized category \mathcal{D} has colimits indexed by I (or Ω_I if you prefer internal category theory)

(H3) For every pullback square

$$\begin{array}{ccc} x & \xrightarrow{\bar{g}} & y' \\ \bar{p} \downarrow & \lrcorner & p \downarrow \\ y & \xrightarrow{g} & z \end{array}$$

in \mathcal{C} with $p, \bar{p} \in \mathcal{C}$, the image under \mathcal{D} is vertically right adjointable

- The parametrized category \mathcal{D} has limits indexed by P

(H4) For every pullback square

$$\begin{array}{ccc} x & \xrightarrow{\bar{i}} & y' \\ \bar{p} \downarrow & \lrcorner & p \downarrow \\ y & \xrightarrow{i} & z \end{array}$$

with $i, \bar{i} \in I$ and $p, \bar{p} \in P$, the image under \mathcal{D} is biadjointable

- Limits indexed by P commute with colimits indexed by I in \mathcal{D}

(H5) Every morphism $f \in I \cap P$ is n -truncated for some $n \in \mathbb{N}$ which may depend on f

We will revisit each of these hypotheses shortly, but let us first mention two universal problems one could consider, and discuss how they are related.

First, note that (H1)–(H5) in fact make sense for a functor

$$\mathcal{D} : \mathcal{C}^{\text{op}} \longrightarrow \mathbb{A}$$

valued in *any* 2-category \mathbb{A} . This may be viewed as an \mathbb{A} -object internal to $\text{PSh}(\mathcal{C})$ and is the perspective taken in [BM24]. We then may ask:

(Q1) Is there an initial 2-category \mathbb{A} such that there exists a \mathcal{C} -parametrized \mathbb{A} -object satisfying (H1)–(H5)? If so, what is it? I.e., is there a 2-category \mathbb{A} along with a functor $b : \mathcal{C}^{\text{op}} \rightarrow \mathbb{A}$ satisfying (H1)–(H5) such that every other functor $b' : \mathcal{C}^{\text{op}} \rightarrow \mathbb{A}'$ satisfying (H1)–(H5) factors as $\mathcal{C}^{\text{op}} \xrightarrow{b} \mathbb{A} \rightarrow \mathbb{A}'$ for a unique 2-functor $\mathbb{A} \rightarrow \mathbb{A}'$?

This is the most natural question to ask as someone interested in six functors, as then, given a pullback functor $\mathcal{D} : \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$ satisfying (H1)–(H5), one may package it fully into a 2-functor $\mathbb{A} \rightarrow \text{Cat}$. Based on intuition from six functor formalisms, one would expect (hope) that the answer to (Q1) is some sort of span 2-category where the left legs encode the pullback functor \mathcal{D} , the right legs encode an exceptional pushforward defined on a class of morphisms E generated under composition by I and P , and 2-morphisms remembering the adjunctions which define the exceptional pushforward for morphisms in I and P .

In a similar vein, the interested parametrized category theorist may ask:

(Q2) What is the free \mathcal{C} -parametrized category (generated by a point, say) satisfying (H1)–(H5)?

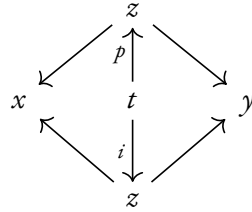
It turns out that these two questions are related as follows:

Theorem 2.1 ([CLL25]). *There exists an initial functor $b : \mathcal{C}^{\text{op}} \rightarrow \mathbb{A}$ satisfying (H1)–(H5) and it is characterized by the property that for every $a \in \mathcal{C}$,*

$$\text{Hom}_{\mathbb{A}}(b(a), b(-)) : \mathcal{C}^{\text{op}} \longrightarrow \text{Cat}$$

is the free \mathcal{C} -parametrized category satisfying (H1)–(H5) generated by $\text{id}_{b(a)}$.

Using this, the current state of the art is the following: Suppose additionally that I and P are closed under taking diagonals. Let $\text{SPAN}_{I,P}(\mathcal{C})$ denote the 2-category whose underlying 1-category is $\text{Span}(\mathcal{C}, E)$ and whose 2-morphisms are given by spans of spans



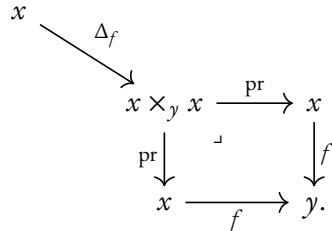
where $p \in P$ and $i \in I$. Then

Theorem 2.2 ([CLL25]). *The inclusion $b : \mathcal{C}^{\text{op}} \rightarrow \text{SPAN}_{I,P}(\mathcal{C})$ as left legs is the initial map satisfying (H1)–(H5).*

3 The norm map and truncation

In this section we look more in depth at the truncation hypothesis (H5). Recall that (H4) asserts that certain parametrized colimits and limits *commute*. However, in defining the exceptional pushforward we need a canonical identification $\text{Nm}_f : f_{\natural} \rightarrow f_*$ for all $f \in I \cap P$. This is a stronger assertion which says that $(I \cap P)$ -indexed colimits *agree* with $(I \cap P)$ -indexed limits.

In classical, unparametrized semi-additivity, if one has that finite coproducts and finite products commute, then one in fact gets for free that they agree by an Eckmann–Hilton type argument. A similar phenomenon occurs in the parametrized setting where the hypothesis (H5) that every $f \in I \cap P$ is n -truncated for some n ensures that the limits and colimits are sufficiently finite for commutation to imply equivalence. We first recall the classical argument for agreement: Let $f : x \rightarrow y \in I \cap P$. Then, we may form the pullback



By (H4), we have a canonical equivalence

$$f_{\natural} \text{pr}_* \xrightarrow{\cong} f_* \text{pr}_{\natural}.$$

Pre-composing both sides with $(\Delta_f)_\natural$ we then get a canonical twisted norm map

$$f_\natural \operatorname{pr}_*(\Delta_f)_\natural \xrightarrow{\cong} f_*.$$

Let us fix the notation $D_f = \operatorname{pr}_*(\Delta_f)_\natural : \mathcal{D}(x) \rightarrow \mathcal{D}(x)$, which we will refer to as the *twist* of f so that we have canonical norm maps

$$f_\natural D_f \xrightarrow{\cong} f_*.$$

Proposition 3.1. *D_f is invertible and there exists a canonical equivalence $D_f^{-1} \simeq D_{\Delta_f}$.*

Proof. By definition, we have that

$$D_f D_{\Delta_f} = \operatorname{pr}_*(\Delta_f)_\natural D_{\Delta_f} \xrightarrow{\cong} \operatorname{pr}_*(\Delta_f)_* = \operatorname{id}.$$

Thus we have that D_f has a right inverse given by D_{Δ_f} .

However, utilizing the same result for f replaced by Δ_f , it follows that we have canonical equivalences

$$D_f \xrightarrow{\cong} D_f D_{\Delta_f} D_{\Delta_{\Delta_f}} \xrightarrow{\cong} D_{\Delta_{\Delta_f}}.$$

Thus we have a canonical equivalence $D_f \simeq D_{\Delta_{\Delta_f}}$. But $D_{\Delta_{\Delta_f}}$ has left inverse given by D_{Δ_f} . It follows that D_{Δ_f} is both a left and right inverse of D_f , as claimed. \square

Corollary 3.2. *If f is n -truncated for some n , then there is a canonical equivalence $D_f \simeq \operatorname{id}$, and hence a canonical equivalence*

$$\operatorname{Nm}_f : f_\natural \xrightarrow{\cong} f_*.$$

Proof. We induct on the truncation level n . If f is an equivalence, i.e. (-1) -truncated then we have an equivalence

$$\begin{array}{ccc} x & \xrightarrow{\Delta_f} & x \times_y x \\ & & \downarrow \operatorname{pr} \\ & & x \xrightarrow{\operatorname{pr}} x \\ & & \downarrow f \\ & & x \xrightarrow{f} y \end{array} \quad \simeq \quad \begin{array}{ccc} x & \xrightarrow{=} & x \\ & & \downarrow f \\ x & \xrightarrow{f} & y \end{array}$$

of diagrams so that $D_f \simeq \operatorname{id}$ by definition.

If f is n -truncated, then Δ_f is $(n - 1)$ -truncated, so the result follows from induction via the canonical equivalence $D_f \simeq D_{\Delta_f}^{-1}$. \square

3.1 *K*-theory!

It turns out that the structure in Proposition 3.1 is actually downstream of a much stronger statement.

Before embarking on this, let us make some restrictions. Rather than constantly working with $I \cap P$, let's assume make the assumption that $I = \mathcal{C}$ and $P = E$. That is, we have a category \mathcal{C}

with finite limits and a wide subcategory $E \subseteq \mathcal{C}$ closed under base-change and taking diagonals. Additionally, we are supposing that we have a parametrized category

$$\mathcal{D} : \mathcal{C}^{\text{op}} \longrightarrow \text{Cat}$$

which

- (i) has colimits indexed by \mathcal{C}
- (ii) limits indexed by E
- (iii) colimits indexed by \mathcal{C} and limits indexed by E commute.

By the earlier discussion, for every $f \in E$ we have a twisted norm map

$$f_{\natural} D_f \longrightarrow f_*$$

and we are interested in studying the twists D_f .

Proposition 3.3. *The twists $f \mapsto D_f$ are functorial in equivalences, i.e. it extends to a map*

$$\begin{aligned} E_{x/y}^{\simeq} &\longrightarrow \text{End}(\mathcal{D}(x)) \\ (f : x \rightarrow y) &\longmapsto D_f \end{aligned}$$

Proof. Suppose that we have a commutative triangle

$$\begin{array}{ccc} & x & \\ f \swarrow & & \searrow g \\ y & \xrightarrow[\alpha]{\simeq} & z. \end{array}$$

Then we have pullback squares

$$\begin{array}{ccccc} & & & & x \\ & & & & \swarrow \Delta_f \simeq \Delta_g \\ & & & & x \times_y x \xrightarrow{\text{pr}} x \\ & & & \lrcorner & \downarrow f \\ & & & \lrcorner & x \xrightarrow{f} y \\ & & & \lrcorner & \downarrow \alpha \\ & & & \lrcorner & x \xrightarrow{g} z. \\ & & & \lrcorner & \end{array}$$

It follows that we have a canonical equivalence $D_g = (\text{id}_x)_* \text{pr}_*(\Delta_f)_{\natural} \simeq \text{pr}_*(\Delta_f)_{\natural} = D_f$. □

We then have the following:

Proposition 3.4. *There exists a commutative triangle*

$$\begin{array}{ccc}
 E_{x/}^{\simeq} & \xrightarrow{D(-)} & \text{End}(\mathcal{D}(x)) \\
 & \searrow \Omega & \nearrow \Sigma(-) \\
 & & K(E_{x/,/x})
 \end{array}$$

where $K(E_{x/,/x}) \rightarrow \text{End}(\mathcal{D}(x))$ is \mathbb{B}_1 . Here,

- (i) $E_{x/,/x}$ is the slice category $(E_{x/})_{/\text{id}_x}$
- (ii) Ω is given by

$$\begin{aligned}
 \Omega : E_{x/} &\longrightarrow E_{x/,/x} \\
 (f : x \rightarrow y) &\longmapsto (x \xrightarrow{\Delta} x \times_y x \xrightarrow{\text{pr}} x).
 \end{aligned}$$

Proof. We construct a map $BK(E_{x/,/x}) \rightarrow \mathcal{D}$ sending $*$ to x which, upon taking loops, gives the required map $K(E_{x/,/x}) \rightarrow \text{End}(\mathcal{D}(x))$. We do this by Waldhausen's S_\bullet -construction of K -theory.

Indeed, $BK(E_{x/,/x})$ is the geometric realization of the simplicial space $S_\bullet(E_{x/,/x})$ where $S_n(E_{x/,/x})$ is the groupoid of diagrams of the form

$$\begin{array}{ccccccccccc}
 x & \xrightarrow{f_{11}} & x_{11} & \xrightarrow{f_{12}} & x_{12} & \xrightarrow{f_{13}} & \cdots & \longrightarrow & x_{1(n-1)} & \xrightarrow{f_{1n}} & x_{1n} \\
 & & \downarrow g_{11} & \lrcorner & \downarrow g_{12} & \lrcorner & & & \downarrow & \lrcorner & \downarrow g_{1n} \\
 & & x & \xrightarrow{f_{22}} & x_{22} & \xrightarrow{f_{23}} & \cdots & \longrightarrow & x_{2(n-1)} & \xrightarrow{f_{2n}} & x_{2n} \\
 & & & & \downarrow g_{22} & \lrcorner & & & \downarrow & \lrcorner & \downarrow g_{2n} \\
 & & & & x & \longrightarrow & \cdots & \longrightarrow & x_{3(n-1)} & \xrightarrow{f_{3n}} & x_{3n} \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & \vdots & & \vdots \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & x & \xrightarrow{f_{nn}} & x_{nn} \\
 & & & & & & & & & & \downarrow g_{nn} \\
 & & & & & & & & & & x
 \end{array} \tag{3.1}$$

where every square consisting of horizontal and vertical arrows is a pullback. Since the geometric realization is left adjoint to the nerve, it suffices to construct a map $S_\bullet(E_{x/,/x}) \rightarrow N(\mathcal{D}^{\simeq})$ of simplicial spaces. (In fact, it suffices to construct a map $S_\bullet(E_{x/,/x}) \rightarrow N(\mathcal{D})$ as $S_\bullet(E_{x/,/x})$ already has “enough inverses” for the associated category to already be a groupoid).

For this, we take the map to be the one that first sends a diagram of the form (3.1) to the diagram

$$\begin{array}{ccccccc}
\mathcal{D}(x) & \xrightarrow{(f_{11})_{\natural}} & \mathcal{D}(x_{11}) & \xrightarrow{(f_{12})_{\natural}} & \mathcal{D}(x_{12}) & \xrightarrow{(f_{13})_{\natural}} & \cdots \longrightarrow \mathcal{D}(x_{1(n-1)}) & \xrightarrow{(f_{1n})_{\natural}} & \mathcal{D}(x_{1n}) \\
& & \downarrow (g_{11})_* & & \downarrow (g_{12})_* & & \downarrow & & \downarrow (g_{1n})_* \\
& & \mathcal{D}(x) & \xrightarrow{(f_{22})_{\natural}} & \mathcal{D}(x_{22}) & \xrightarrow{(f_{23})_{\natural}} & \cdots \longrightarrow \mathcal{D}(x_{2(n-1)}) & \xrightarrow{(f_{2n})_{\natural}} & \mathcal{D}(x_{2n}) \\
& & & & \downarrow (g_{22})_* & & \downarrow & & \downarrow (g_{2n})_* \\
& & & & \mathcal{D}(x) & \longrightarrow & \cdots \longrightarrow \mathcal{D}(x_{3(n-1)}) & \xrightarrow{(f_{3n})_{\natural}} & \mathcal{D}(x_{3n}) \\
& & & & & & \downarrow & & \downarrow \\
& & & & & & \vdots & & \vdots \\
& & & & & & \downarrow & & \downarrow \\
& & & & & & \mathcal{D}(x) & \xrightarrow{(f_{nn})_{\natural}} & \mathcal{D}(x_{nn}) \\
& & & & & & & & \downarrow (g_{nn})_* \\
& & & & & & & & \mathcal{D}(x)
\end{array}$$

where the commutative squares come from the double Beck–Chevalley condition.

This simplicial space then maps to $N(\mathcal{D})$ by restricting to the diagonal. It is then clear that one gets the desired factorization of $D(-)$ from the definition of $D(-)$. \square

Corollary 3.5. *If each $f \in E$ is n -truncated for some n , then $K(E_{x/,/x}) \simeq 0$. In particular, each $D_f \simeq \text{id}$.*

Proof. We have that looping $\Omega : E_{x/,/x} \rightarrow E_{x/,/x}$ corresponds to taking the diagonal. However, on K -theory, Ω induces multiplication by -1 . Thus

$$K(E_{x/,/x}) = \text{colim} \left(K(E_{x/,/x}) \xrightarrow{-\Omega} K(E_{x/,/x}) \xrightarrow{-\Omega} \cdots \right).$$

However, for every $x \rightarrow e \rightarrow x \in K(E_{x/,/x})$ we have that $\Omega^n e = 0$ for some n by assumption. Thus $K(E_{x/,/x}) \simeq 0$. \square

These K -theory spaces are not unfamiliar, either.

Example 3.6 ([DK24, Proposition 4.12]). Let \mathcal{C} be the opposite category of \mathbb{E}_∞ -algebras over some \mathbb{E}_∞ -ring R and let E be the class of morphisms of affine schemes over $\text{Spec}(R)$ of finite presentation. Then

$$K(E_{\text{Spec } S/,/\text{Spec } S}) \simeq K(S).$$

Indeed, first note that

$$K(E_{x/,/x}) \simeq K(\text{Stab}(E_{x/,/x}))$$

so that the K -theory may be computed after stabilization. Moreover,

$$\text{Stab}(\text{CAlg}_{S/,/S}^{\text{f.p.,op}}) \simeq \text{Stab}(\text{CAlg}_{S'}^{\text{f.p.,op}})$$

since $\text{Stab}(\mathcal{D}) \simeq \text{Stab}(\mathcal{D}_{*/})$ for any category with a terminal object $*$. Thus it suffices to identify $\text{Stab}(\text{CAlg}_{S'}^{\text{f.p.,op}})$ with $\text{Mod}_S^{\text{perf}}$. By [Lur17, Theorem 7.3.4.13],

$$\text{Stab}(\text{CAlg}_{S'}) \simeq \text{Mod}_S.$$

Now, if \mathcal{D} is a compactly generated category, one has that

$$\text{Stab}(\mathcal{D}) \simeq \text{Ind}(\text{Stab}^{\Sigma}(\mathcal{D}^{\omega})) \simeq \text{Ind}(\text{Stab}((\mathcal{D}^{\omega})^{\text{op}})^{\text{op}}).$$

Thus

$$\text{Mod}_S \simeq \text{Stab}(\text{CAlg}_{S'}) \simeq \text{Ind}(\text{Stab}(\text{CAlg}_{S'}^{\text{f.p.,op}})^{\text{op}}).$$

Taking compact objects and noting that $\text{Stab}(\text{CAlg}_{S'}^{\text{f.p.,op}})^{\text{op}}$ is idempotent complete, we get that

$$\text{Mod}_S^{\text{perf}} \simeq \text{Stab}(\text{CAlg}_{S'}^{\text{f.p.,op}})^{\text{op}}.$$

But $\text{Mod}_S^{\text{perf,op}} \simeq \text{Mod}_S^{\text{perf}}$ by taking duals so by taking opposites of the above we get that

$$\text{Mod}_S^{\text{perf}} \simeq \text{Stab}(\text{CAlg}_{S'}^{\text{f.p.,op}})$$

as required.

We now briefly discuss interpretations. One should think of $K(E_{x/},/x)$ as an abstract notion of virtual vector bundles over an object $x \in \mathcal{C}$. Indeed, an object is given by

$$x \xrightarrow{s} e \xrightarrow{p} x$$

with $ps \simeq \text{id}_x$. Here, e should be thought of as the total space of a vector bundle over x , s the zero section, and p the projection map. The twist of the virtual vector bundle $[e] \in K(E_{x/},/x)$ is then given by $p_*s_! : \mathcal{D}(x) \rightarrow \mathcal{D}(x)$ which should be thought of in analogy with the push-pull formula for computing the Thom spectrum of a vector bundle. Moreover, the twist of a morphism $f : x \rightarrow y$ is given by the automorphism associated to the “vector bundle”

$$x \xrightarrow{\Delta_f} x \times_y x \xrightarrow{\text{pr}} x$$

It is geometric (somewhat) folklore that in derived algebraic geometry, HKR gives an equivalence between the formal neighborhood of the diagonal in $x \times_y x$ and the total space of the relative tangent bundle T_f . Thus, at least notationally, we should denote the class $[x \times_y x] \in K(E_{x/},/x)$ by $[T_f]$ and think of it as the relative tangent bundle of f . As a consequence of this, we should think of $-[x \times_y x]$ as being a model for the relative cotangent bundle L_f . It turns out that this notation is compatible with the isomorphism in Example 3.6.

Then, the twisted norm maps give a canonical equivalence

$$f_! \Sigma^{T_f} \longrightarrow f_*.$$

Such relationships are ubiquitous in mathematics with six functors and show up in literature discussing Wirthmüller contexts and twisted ambidexterity. See for example [MSo6, Cno23, Hoy17].

4 A universal statement

We said earlier that a functor $\mathcal{D} : \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$ satisfying (H1)–(H4) (with $I = \mathcal{C}$ and $P = E$) has induced twist functors

$$\Sigma^{(-)} : K(E_{x/,/x}) \longrightarrow \text{End}(\mathcal{D}(x))$$

for every $x \in \mathcal{C}$. Now, these K -theory classes assemble into a functor

$$\begin{aligned} K : \mathcal{C}^{\text{op}} &\longrightarrow \text{Sp} \\ x &\longmapsto K(E_{x/,/x}) \end{aligned}$$

where the transition maps along $f : x \rightarrow y$ are given by K -theory of the basechange along f map $E_{y/,/y} \rightarrow E_{x/,/x}$.

Definition 4.1. Denote by $\text{Tw}(\mathcal{C}, E)$ the Cartesian unstraightening of the functor

$$\begin{aligned} BK : \mathcal{C}^{\text{op}} &\longrightarrow \text{Sp} \\ x &\longmapsto BK(E_{x/,/x}) \end{aligned}$$

Morally, $\text{Tw}(\mathcal{C}, E)$ has the same objects as \mathcal{C} , and morphisms $x \rightarrow y$ are pairs (f, ξ) where $f : x \rightarrow y$ is a morphism in \mathcal{C} and $\xi \in K(E_{x/,/x})$. Composition is given by

$$x \xrightarrow{(f, \xi)} y \xrightarrow{(g, \eta)} z = x \xrightarrow{(gf, \xi + f^* \eta)} z.$$

The twist functors $\Sigma^{(-)}$ in fact upgrade to define a functor

$$\begin{aligned} \text{Tw}(\mathcal{C}, E) &\longrightarrow \text{Cat} \\ x &\longmapsto \mathcal{D}(x) \\ (f, \xi) &\longmapsto f_* \Sigma^\xi. \end{aligned} \tag{4.1}$$

There are then two sections

$$\begin{aligned} \mathcal{C} &\longrightarrow \text{Tw}(\mathcal{C}, E) \\ x &\longmapsto x \\ f &\longmapsto (f, 0) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} E &\longrightarrow \text{Tw}(\mathcal{C}, E) \\ x &\longmapsto x \\ f &\longmapsto (f, \mathcal{L}_f). \end{aligned} \tag{4.3}$$

Composing these two sections with (4.1) then recover the two functors $(-)_* : \mathcal{C} \rightarrow \text{Cat}$ and $(-)_\natural : E \rightarrow \text{Cat}$, respectively.

This phenomenon, it turns out, is purely a consequence of diagrammatics involving pullbacks in \mathcal{C} and captures all the information about $(-)_*$ and $(-)_\natural$, including their double Beck–Chevalley relationship.

Theorem 4.2 (F.). $\text{Tw}(\mathcal{C}, E)$ is the category freely generated by a copy of \mathcal{C} and E , modulo relations generated by pullbacks of the form

$$\begin{array}{ccc} x & \xrightarrow{\tilde{f}} & y' \\ \tilde{g} \downarrow & \lrcorner & \downarrow g \\ y & \xrightarrow{f} & z \end{array}$$

with $g, \tilde{g} \in E$. Under this description, the maps $E \rightarrow \text{Tw}(\mathcal{C}, E)$ and $\mathcal{C} \rightarrow \text{Tw}(\mathcal{C}, E)$ correspond to (4.3) and (4.2).

Remark 4.3. Here it is crucial that $E \subseteq \mathcal{C}$. One may instead try to consider the analogous, more general statement for the category freely generated by subcategories I, P of \mathcal{C} modulo pullback squares in \mathcal{C} relating I and P (the case above then being $I = \mathcal{C}$ and $E = P$). This analogous statement is false.

5 Application: Coherent norm maps

Let $F : \mathcal{C}^{\text{op}} \rightarrow \mathbb{A}$ be a functor satisfying (H1)–(H4) with $I = P = E = \mathcal{C}$. Then we obtain two functors

$$F^L, F^R : \text{Span}(\mathcal{C}) \rightarrow \mathbb{A}$$

where F^L is given by F on left leg spans and by the left adjoints on right leg spans, with higher cells given by Beck–Chevalley squares. Similarly for F^R but with right adjoints.

Definition 5.1. A *coherent norm map* on F is an equivalence $\alpha : F^L \simeq F^R$ of functors.

What sort of data does such an equivalence α contain? When restricted to $\mathcal{C}^{\text{op}} \subseteq \text{Span}(\mathcal{C})$ it gives an auto-equivalence $\alpha|_{\mathcal{C}^{\text{op}}} : F \simeq F$ of F . The space of all coherent norm maps deformation retracts onto the subspace of those for which this auto-equivalence is the identity, so we may assume $\alpha|_{\mathcal{C}^{\text{op}}}$ is the identity.

Then, restricting to right legs, we obtain an equivalence $\alpha|_e : F^L|_e \simeq F^R|_e$. For every $f : x \rightarrow y$ in \mathcal{C} we therefore obtain a commutative square

$$\begin{array}{ccc} F(x) & \xrightarrow{\alpha_x} & F(x) \\ F(f)^L \downarrow & & \downarrow F(f)^R \\ F(y) & \xrightarrow{\alpha_y} & F(y). \end{array}$$

By the assumption that $\alpha|_{\mathcal{C}^{\text{op}}}$ is the identity, we have that $\alpha_x = \text{id}_{F(x)}$ and $\alpha_y = \text{id}_{F(y)}$, so the data of such a square is simply an equivalence

$$\text{Nm}_f : F(f)^L \xrightarrow{\simeq} F(f)^R.$$

The higher cells in $\text{Span}(\mathcal{C})$ then encode compatibilities of each Nm_f with certain Beck–Chevalley squares. Indeed, suppose that we have a pullback square

$$\begin{array}{ccc} x & \xrightarrow{\tilde{g}} & y' \\ \tilde{f} \downarrow & \lrcorner & f \downarrow \\ y & \xrightarrow{g} & z. \end{array}$$

The corresponding composition in $\text{Span}(\mathcal{C})$ implies a commutative cube

$$\begin{array}{ccccc}
 & & & F(x) & \xleftarrow{F(\bar{g})} & F(y') \\
 & & \nearrow \alpha_x & \downarrow F(\tilde{f})^R & \nearrow \alpha_{y'} & \downarrow F(f)^R \\
 F(x) & \xleftarrow{F(\bar{g})} & F(y') & & & \\
 \downarrow F(\tilde{f})^L & & \downarrow F(f)^L & & & \\
 F(y) & \xleftarrow{F(g)} & F(z) & \xleftarrow{F(g)} & & F(z) \\
 & \nearrow \alpha_y & \nearrow \alpha_z & & &
 \end{array}$$

Utilizing again the assumption that $\alpha|_{\text{op}}$ is the identity, this commutative cube amounts to the data of an equivalence

$$\begin{array}{ccc}
 F(x) \xleftarrow{F(\bar{g})} F(y') & & F(x) \xleftarrow{F(\bar{g})} F(y') \\
 \downarrow F(\tilde{f})^L & \text{Nm}_{\tilde{f}} & \downarrow F(f)^L \\
 F(y) \xleftarrow{F(g)} F(z) & \xleftarrow{F(\tilde{f})^R} & F(y) \xleftarrow{F(g)} F(z) \\
 & & \downarrow F(f)^R \\
 & & F(y) \xleftarrow{F(g)} F(z)
 \end{array}$$

where both squares are filled with the Beck–Chevalley equivalence. Higher cells encode further coherence conditions.

Theorem 5.2 (F). *The space of coherent norm maps is equivalent to the space of trivializations of the twists of F , i.e. the space of factorizations of the induced map*

$$\text{Tw}(\mathcal{C}, \mathcal{C}) \longrightarrow \text{Cat}$$

through the projection $\text{Tw}(\mathcal{C}, \mathcal{C}) \rightarrow \mathcal{C}$.

In particular, if all maps are n -truncated for some n so that $\text{Tw}(\mathcal{C}, \mathcal{C}) \xrightarrow{\cong} \mathcal{C}$ is an equivalence, then the space of coherent norm maps is contractible.

This in the standard setting of semi-additivity/ambidexterity where the truncation hypothesis is assumed, there is a unique coherent norm map extending the norm map seen in literature. For geometers this is relevant as it allows one to perform descent on norm-like equivalences.

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