Qualifying Exam Syllabus

Will Fisher

Committee. David Nadler (Advisor), Constantin Teleman (Chair), Christian Gaetz, Fraydoun Rezakhanlou.

Location. Evans 762 at 2pm, December 11th, 2024.

1. Major Topic: Algebraic Geometry (Algebra)

References. Hartshorne, *Algebraic Geometry* [Ha].

- *Sheaf theory:* Presheaves, sheaves, sheafification; $\mathcal{O}_X$-modules, adjunction of $\widetilde{(-)}$, quasi-coherent and coherent sheaves; pushforward, pullback; flasque sheaves; existence of injective resolutions
- *Scheme theory:* Proj and Spec constructions; adjunction of Γ; proper, (quasi-)separated, finite, finite-type, affine, flat morphisms; closed and open subschemes; fibre products; properties preserved by base change
- *Divisors:* Weil and Cartier divisors; Picard group; excision sequence
- *Cohomology:* Sheaf cohomology; Derived functors; Čech cohomology; Grothendieck vanishing; vanishing of cohomology of quasi-coherent sheaves on Noetherian affines; computation of $H^i(\mathbb{P}^n, \mathcal{O}(m))$; Serre duality (statement)
- *Curves:* Riemann-Roch theorem; Hurwitz's theorem; Pic^0 of elliptic curve; canonical embedding; embeddings into projective space

2. Major Topic: Category Theory (Algebra)

References. Emily Riehl, *Category Theory in Context* [Ri], Cisinski, *Higher Categories and Homotopical Algebra* [Ci], Lurie, *Higher Algebra* [Lu].

- *Basics:* Categories and functors; Yoneda embedding; (co)limits; adjunctions [Ri]
- *Major theorems:* General adjoint functor theorem, Special adjoint functor theorem, Freyd's representability criterion; adjoints in presentable categories; Brown representability (statement) [Ri, Ci]
- *Monads:* Monads; Kleisli category, Eilenberg–Moore category; Barr–Beck/monadicity theorem [Ri]
- *Basics of ∞-categories:* Model categories, derived functors; Kan complexes; limits and colimits; stable categories; presentable and accessible categories; Dold–Kan and dg-models; t-structures, truncation [Ci, Lu]

3. Minor Topic: Algebraic Topology (Geometry/Topology)

References. Hatcher, *Algebraic Topology* [Ha].

- *Homotopy theory:* Homotopy groups; Covering spaces; Van–Kampen's theorem; LES of fibration; Eilenberg–MacLane spaces; Whitehead's theorem
- *(Co)homology:* Ordinary (co)homology; Cellular (co)homology; Poincaré duality; Mayer–Vietoris sequence, LES of pair, excision; UCT; Künneth formula

Qualifying Exam Transcript

Will Fisher

Committee. David Nadler [N], Constantin Teleman [T], Christian Gaetz [G], Fraydoun Rezakhanlou [R].

Disclaimer. This transcript was written from memory after the fact. While I believe it to be close to reality, there is no guarantee of accuracy.

1. Algebraic Geometry (∼45 min)

G: What is a morphism of schemes?

Me: A scheme is a locally ringed space, i.e. a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings. A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f^{\mathrm{top}} : X \to Y$ of topological spaces along with a morphism of sheaves $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ such that each induced map on stalks is a morphism of local rings.

I forgot to add that every stalk of the structure sheaf should be a local ring in the definition of a locally ringed space.

G: Consider the morphism from the hyperbola down to the affine line—do you understand the picture I'm describing?

Me: Yes.

G: Your syllabus mentions all these adjectives "finite, finite type, proper, separated." Can you write down this morphism and explain which of these properties hold for this morphism?

I draw the picture (over $\mathbb{R}$) of the hyperbola in $\mathbb{A}^2$ with projection down to the x-axis as well as the properties I am to address.

Me: Immediately I know we are not proper because the image is $\mathbb{A}^1 \setminus 0$ which is not closed.

G: Can you tell me what proper means?

Me: Proper means finite type, separated and universally closed. Universally closed means that the underlying topological map is closed and that it remains closed under any base change. So here I guess I'm using that the morphism isn't even closed.

I start to talk about being finite type before being interrupted.

G: Before you continue—is there any property you can immediately rule out by not being proper?

Me: Yes. We cannot be finite since being finite is equivalent to being proper and quasi-finite. Also, the induced map on rings is $k[x] \hookrightarrow k[x, y]/(xy - 1)$ which is finitely generated by y as a $k[x]$-algebra, so we are finite type. Finally, we're also separated because any morphism of affines is separated. One can see this by explicitly writing out the diagonal and checking that it is a closed immersion.

N: What happens if we take the projective closure of this hyperbola? What do we get?

Me: The parts going off to infinity should get identified. Either they will all close up or I will get two pieces. I'm not sure which will get identified though without working it out.

N: Do you have a guess?

Me: Not that I'm confident enough to state. But I can work it out.

N: Sure. Let's do that.

Me: So the projective closure is given by homogenizing the equation, so it's given by $V(xy-z^2) \subseteq \mathbb{P}^2$. This is a quadric so its determined by the rank of the quadratic form.

I sat contemplating whether there is another way to do this, not wanting to write down any matrices.

Me: Oh. This is also a hypersurface in $\mathbb{P}^2$ so by the degree genus formula if it is smooth then it is $\mathbb{P}^1$.

N: Is it?

Me: I can check. So we can check in each chart and use the Jacobian criteria which in codimension one amounts to checking that the gradient of the defining function is never zero on the curve.

N: Before you do that, is there an easier way to do this?

Me: Um, I'm not sure.

N: What happens if you view this equation in $\mathbb{A}^3$?

Me: I get the projective cone.

N: How does the smoothness of this relate to the smoothness of your projective variety?

Me: Well it is not going to be smooth at the cone point, but if I take a slice of the cone I get my original variety so I suspect smoothness away from the cone point implies smoothness of my original variety.

N: I'm not sure what you mean by this "slice" business.

Me: Well if I blow up the cone at the cone point, I get $\mathbb{P}^1$ times my original variety so I suppose that shows that smoothness away from the cone point is equivalent to smoothness of my original variety.

N: I don't think you mean $\mathbb{P}^1$, but that's alright. Can you go ahead and show smoothness then?

I proceed to compute the gradient of $xy - z^2$ and see that it only vanishes at $(0,0,0)$, i.e. the cone point, so I conclude smoothness of the original projective variety and thus by degree genus that it is indeed $\mathbb{P}^1$.

N: Okay, good. What points do we get at infinity then?

Me: Well, the points at infinity are those with $z = 0$ so we get exactly $[0:1:0]$ and $[1:0:0]$.

N: Can you label this behavior on your original picture?

I sloppily say that the parts of my hyperbola going of to infinity do indeed get identified forming one connected piece, not being careful about which get identified with which.

N: Can you be more precise about which points get identified with which? Maybe label them, like with a and b.

Me: Well I have one point at infinity with $x = 0$, so the two pieces asymptoting to the x-axis will get identified with this point, and the two pieces asymptoting to the y-axis will get identified with $[1 : 0 : 0]$.

N: Okay, good.

T: What if I were to give you a more general equation in $\mathbb{P}^2$, so just any polynomial. Can you tell me about this case? What geometric things can you tell me?

Me: Well, if we're smooth—actually as long as we're just irreducible—the degree genus formula tells us that the arithmetic genus is given by $\binom{d-1}{2}$.

T: Do you know how to show that?

Me: I think the best way is just to do Čech cohomology using the standard affine cover on $\mathbb{P}^2$. Do you want me to try to do that?

N: I'm happy to not spend time on that, unless Teleman you want to see it.

T: No I think that's fine.

N: Going back to your hyperbola, what happens if I also give you the equation $x^2 + y^2 = 2$. What do the two pictures look like? Do they intersect?

Me: Well, by Bezout, we know that as long as these don't share a component, which they don't, then when counted with multiplicity they intersect at four points.

N: What do you mean by multiplicity here?

Me: Well, if we intersect transversally, then the multiplicity is one. If we're tangent, then it's at least two and its essentially measuring the order of tangency. If you want to do it algebraically, its the length of the local ring modded out by our two equations.

N: Can you figure out what happens for the two equations I gave you? Maybe draw it on the picture.

I draw this circle as intersecting at four distinct points without David or me realizing this is exactly the circle which is tangent at two points to the hyperbola.

N: What if I were to change the circle? What other cases could I get?

Me: Well, there's going to be an extremal case when the circle becomes tangent to the hyperbola at two points.

N: Can you figure it out.

Trying to go too fast, I mistakenly label the closest point on the hyperbola to the origin as $(1/\sqrt{2}, 1/\sqrt{2})$.

N: Hmm... What's the equation of the hyperbola?

Me: Oh, oops. I guess the point should be $(1, 1)$. So its a circle of radius $\sqrt{2}$ which I guess actually means the previous picture is wrong and should be this scenario.

We joke about both having had this wrong.

N: Do you know the scheme theoretic intersection here?

Me: Well, its the fiber product.

N: Can you tell me what it is if you know it?

Me: Well, its going to be supported on two points and the stalks will be length two modules.

N: Can you tell me what those could look like?

Me: It should be something that remembers the point and a tangent vector, so something with an embedded point.

I spend a while thinking.

Me: I suppose it will look like the dual numbers, so like $k[x]/(x^2)$ but shifted to the actual point of intersection not the origin.

N: Are there any other possibilities?

I read into this as me missing a possibility and spent a long time thinking, eventually proposing $k[x,y]/(x^2, xy, y^2)$ as another potential option, saying something about us being two dimensional (even though we were not).

N: What the the dimension, or length—I don't know—of what you just wrote?

Me: It's dimension three.

N: Right. So is it a possibility?

Me: No. I'm not sure there are any other ones.

N: I agree. I think $k[x]/(x^2)$ is the only option.

There's a lull in conversation and David eventually asks if anyone has something they would like to ask.

T: Can you tell me about Cartier and Weil divisors? Is there a comparison map in one direction, when are they the same, etc...

Me: Sure. I am going to do this in the level of generality of Hartshorne, even though I know there is potentially a more general setting in which Weil divisors make sense.

I proceed to write down the conditions in Hartshorne under which Weil divisors make sense, then define them, talking about principal divisors and the class group. I then move to defining Cartier divisors, but start to space when I reach what the precise definition of what a principal Cartier is.

Me: Okay, I forget exactly what the principal Cartier divisors are... Maybe I'll just tell the rest of the story and come back to this.

I write $\Gamma(X, \cdots)$ on the board for principal Cartier divisors because I know that they are global sections of some sheaf I am forgetting on the spot.

N: I am confused what's happening right now.

Me: There's a notion of principal Cartier divisors just like there is a notion of principal Weil divisors, I'm just forgetting the precise definition at the moment... Aha! I remember, they are the image of $\Gamma(X, \mathcal{K}^*)$, i.e. those which have a globally defining function.

I continue with the overview, describing the comparison map which associates to each Cartier divisor a Weil divisor. I mention that this map respects principal divisors and that in general it is neither injective nor surjective, but mention that I think the examples of non-injectivity are not relevant in the level of generality Hartshorne considers. I then draw the conic to start describing a failure of surjectivity, but unlike the conic from the start of the exam I draw this one sideways...

N: This looks a lot like the cone you drew earlier but now you've drawn it sideways! Is it the same cone?

Me: Yes.

N: What is the equation?

Me: The same equation $xy - z^2$, just maybe turn your head sideways. Anyways, any Weil divisor coming from a Cartier divisor is locally principal so we can look at a ruling on this cone. At the cone point the cone is singular, so the Zariski tangent space is higher than two dimensions. Since everything here is sufficiently nice, i.e. normal and Cohen–Macauly, regular sequences work how we expect so we can't cut out a one dimensional thing with a single equation if the tangent space is > 2 dimensional.

N: How do line bundles fit into this?

Me: Well for every scheme X we can define $\operatorname{Pic}(X)$ which are invertible sheaves on X up to isomorphism with the group operation taking tensor products. Then we have a morphism from Cartier divisors to $\operatorname{Pic}(X)$ by looking at the subsheaf of $\mathcal{K}^*$ generated by the local defining functions. If X is integral, and maybe also Noetherian, though that's a running assumption in Hartshorne, then this map is an isomorphism.

T: What is Pic of projective space, say of any dimension?

Me: It is $\mathbb{Z}$, with the isomorphism given by degree.

N: Can you tell me what this degree map is. What if we have something like $\mathbb{P}^1 \times \mathbb{P}^1$, say?

Me: Sure. First, $\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ which is given explicitly by $\operatorname{pr}_1^* \mathbb{P}^1 \oplus \operatorname{pr}_2^* \mathbb{P}^1$.

N: Maybe you want a Pic there? I'm not sure.

Me: Oh duh, I meant to write $\operatorname{pr}_1^* \operatorname{Pic} \mathbb{P}^1 \oplus \operatorname{pr}_2^* \operatorname{Pic} \mathbb{P}^1$.

N: Okay so what's all this degree stuff.

Me: Well, in this setting all three notions agree so the easiest definition is just the sum of the coefficients of a representative of the corresponding divisor class.

N: I'm not sure I follow.

There then ensued a long back and forth where it felt like David was probing for a specific definition of degree but not clarifying which, or what was wrong with the one I gave. I feared that he wanted one that only referenced line bundles, but I was spacing on the spot and trying to avoid this. I gave as the next alternative the Hilbert polynomial of a coherent sheaf on projective space and how to read off the degree from the lead coefficient. I also mentioned that for an effective divisor I can view it as the number of points in the intersection of the divisor with its dimension many generic hyperplanes. Finally, I move to saying the right thing...

Me: I guess what I mean by the hyperplane thing is that if we take a copy of $\mathbb{P}^1$ inside $\mathbb{P}^n$, then I'm looking at the degree of the line bundle when pulled back to this $\mathbb{P}^1$ using $\operatorname{Pic}\mathbb{P}^1 \cong \mathbb{Z}$.

I then make a wavy gesture with my hand to indicate an embedded $\mathbb{P}^1$*.*

N: I'm happy with that. Though you did this motion with your hand when describing the copy of $\mathbb{P}^1$ is that what you mean?

Me: Oops, no I mean a linear copy of $\mathbb{P}^1$ inside $\mathbb{P}^n$, so I should've made a straight movement.

N: How would you define the degree on say a curve that's not $\mathbb{P}^1$?

Me: I would define it as the sum of the coefficients of the corresponding divisor.

N: Hmmm... Okay so I guess curves are just the fundamental notion for you. Interesting.

N: How's does this work in the $\mathbb{P}^1 \times \mathbb{P}^1$ example?

Me: I'm looking at the degrees when restricted to vertical and horizontal copies of $\mathbb{P}^1$.

2. Category Theory (∼45 min)

N: I gotta say, I was happy to see a category theory major with some actual theorems not just all the basics... Can you give me a version of Barr–Beck? I'm happy if you want to give a sufficient or weaker criterion if that's the one you learned.

Me: Do you want me to explain the surrounding definitions and story or just the statement?

N: Just the statement is fine.

I write down Barr–Beck and give one version that says U *creates split coequalizers and one version that says lifts of split coequalizers exist and that* U *is conservative. I'm about to write down the Crude Monadicity Theorem as well before getting stopped.*

N: What do these words mean? What do you mean by monadic?

I give the definition of monadic, mentioning that $\mathrm{Mod}_T(\mathcal{C})$ *is final in the category of categories with adjuncitons inducing* T *so that we get an induced map* $\mathcal{D} \to \mathrm{Mod}_T(\mathcal{C})$*.*

Me: So being monadic asks that this map is an equivalence. There's also a version which says when this is an isomorphism instead of just an equivalence.

N: Oh I haven't thought of that, maybe you can teach me.

Me: Yeah, I mean I think there's not much reason to care about that.

N: Okay, this is great. Now suppose I have a map $f : X \to Y$ of schemes, can you tell me when the pushforward on quasi-coherent sheaves is monadic?

I write down on the board what is being asked of me.

Me: Hmmm... Do you mean monadic or whether the pullback is comonadic? I know that is what most relates to descent in algebraic geometry.

N: I want you to do it for pushforward. So f_* would be U in your notation.

Me: Okay, sure. And I guess I should suppose that f is something like qcqs here so that pushforward actually lands in $\mathrm{QCoh}(Y)$?

N: Yeah, sure. The point isn't to be a technical AG problem.

I spend some time thinking.

Me: Okay let me try it for affines first. If $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, then f_* sends an A-module M to M_B, its restriction of scalars. I'm not sure if it's conservative, really ever, though, since being isomorphic as B-modules is easier than being isomorphic as A-modules.

N: Well I think there is some confusion here. Conservativity is about a map becoming an isomorphism after applying your functor.

Me: Oh of course, so we're always conservative since being an isomorphism of modules can be checked on the level of sets.

I write down a picture like $M_B \rightrightarrows N_B \to L$ representing a split coequalizer to start verifying the second condition of Barr–Beck. I spend a while thinking before David interjects.

N: Well first maybe you should tell me what the monad even is here.

Me: Pullback is given by tensoring with A so the monad is $T = (M \otimes_B A)_B$.

N: I think maybe what you've written is TM not T.

Me: Oh true.

N: Now can you tell me what a module for this monad is?

Me: Well we have to give an action map $TM \xrightarrow{a} M$, so a B-linear map $(M \otimes_B A)_B \to M$ which amounts to a B-bilinear map $M \times A \to M$. So I guess I'm essentially giving M the structure of an A-module which extends its B-module structure. So depending on if this extension is unique then I think it is monadic. Maybe if $B \to A$ is injective then this extension is unique so we're monadic.

N: Can you write down the map from A-modules to T-modules?

Me: Yeah... So it should send an A-module to itself where the underlying B-module is the restriction of scalars. Oh wait, I suppose once I've equipped the B-module with an A-module structure extending B then the B-module is uniquely given by restriction of scalars. So it is an equivalence.

T: You mean when $B \to A$ is injective no?

Me: No when I say extends the action of B I mean that it agrees with the restriction of scalars, but that's uniquely determined once I've fixed the A-module structure.

T: Ah okay.

N: Okay so we've learned that it's monadic for any morphism of affines. What about in general? What about, say, if we're projective.

Me: Do you mean like if X is a projective Y scheme, or just like both are projective?

N: Well maybe first can you give me an example where it's not monadic.

Me: I feel like something like $\mathbb{P}^1 \to \operatorname{Spec} k$ will not be because we just lose too much information.

N: Okay.

Me: So pushforward here is just taking global sections, so I don't think we're even conservative. For example $\mathcal{O}(-1) \leftarrow \mathcal{O}(-2)$ both have no global sections but are not isomorphic.

N: And what's the map here?

Me: Oh, right, uh...

I erase what I've written worrying that I potentially don't have a map.

N: No, no. Don't erase it. I just want to know the map.

I put it back on the board, this time with an arrow $\mathcal{O}(-2) \to \mathcal{O}(-1)$ *since that is the direction I can give a map in.*

Me: Just multiplication by the first projective coordinate x_0 should work.

N: Okay, and I'm just curious do you have a guess for what the general answer might be? This is bordering now on an AG problem so it's fine if not.

Me: Oh. Actually I'm now remembering a Hartshorne problem where if f is affine, then modules on X correspond to $f_*\mathcal{O}_X$-algebras via relative Spec. I imagine if I unpack that its essentially what we're doing here.

N: Yeah I agree. You don't need to unpack that.

There's some silence while I wait for a new question.

N: I'm looking at your syllabus. I'm seeing this Kleisli category; I don't even know what that is!

Me: Oh the Eilenberg–Moore category is just the category of T-modules which is final in categories inducing the monad and the Kleisli category is the subcategory of free T-modules and it is initial among those inducing the monad.

T: I see on your syllabus that you've written dg-models. Can you explain this to me? Are you working in the k-linear setting, or?

Me: Maybe I was unclear on the syllabus but what I mean by that is Dold–Kan and how it lets you use chain complexes to do simplicial derived methods. I can talk about that.

T: Sure.

I write the Dold–Kan theorem as it shows up in Higher Algebra, *mentioning as well that if* $\mathcal{A} = \mathrm{Ab}$ *then the equivalence is also a Quillen equivalence.*

Me: What this let us do is— well one can show that this correspondence is also lax monoidal so it gives an identification between algebra objects. So if I want to do derived AG with simplicial commutative rings then I can just as well do it with the homotopy theory of dg-algebras.

T: Do you mean commutative dg-algebras?

Me: Yeah commutative dgas.

N: We'll you don't strictly need commutativity, it's just what most people care about.

Me: Yeah sure. For my example though about simplicial commutative rings it corresponds to commutative dgas.

T: Are you forgetting something? Maybe something about $\mathbb{Q}$.

Me: Oh yeah. I need to be characteristic zero for this.

T: My hero.

N: I see you have t-structures on your syllabus, can you tell me about that without being too technical.

Me: Yeah sure. So morally a t-structure is supposed to capture the idea of homological concentration that we have for chain complexes...

I write down that we're in the setting of $\mathcal{C}$ a stable ∞-category and so $\mathrm{h}\mathcal{C}$ is triangulated, and a t-structure on $\mathcal{C}$ is a t-structure on $\mathrm{h}\mathcal{C}$. I then start to explain what a t-structure is on a triangulated category before being interrupted.

N: I like how you started with an ∞-category. For most people this is a structure on a triangulated category...

Me: Oh yeah. Well I mean I didn't know what a triangulated category was before learning about this for ∞-categories.

N: No, no. I'm not saying it's a bad thing. I actually think it is good. I suppose this is the 21st century after all.

I go on to explain the axioms of a t-structure, and I know there's some third bullet point axiom I'm forgetting but its the arguably least important one...

Me: There's one last technical criterion I think I'm forgetting.

N: That's fine, the point isn't to be overly technical. Can you tell me about the heart?

Me: Yeah. The heart is $\mathcal{C}^\heartsuit = \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$ and it's equivalent to an ordinary category. That's because for $i > 0$ and $X, Y \in \mathcal{C}^\heartsuit$ we have

$$\begin{aligned}\pi_i \mathrm{map}(X, Y) &= \pi_0 \mathrm{map}(\Sigma^i X, Y) \\ &= 0\end{aligned}$$

since $\Sigma^i X$ and Y are shifted off of each other.

N: Great. So what is the derived category of the heart? Is there any maps from it into any categories you've written?

Me: Yeah. So I guess I should add the heart is an abelian category. And for any abelian category $\mathcal{A}$ we can define the right boundedd derived category $\mathcal{D}^-(\mathcal{A}) = \mathrm{Ch}_{\geq 0}(\mathcal{A})[W^{-1}]$ where I'm taking the ∞-categorical localization here.

T: Mmm... Is that non-negative degree chain complexes?

Me: Yeah. So like stuff like

$$\cdots \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0 \longrightarrow 0$$

where we don't have terms far enough to the right.

T: I'm not sure about that... I think—

Me: Oh right! I should just have right bounded complexes so $\mathcal{D}^-(\mathcal{A}) = \mathrm{Ch}^-(\mathcal{A})[W^{-1}]$. Now $\mathcal{D}^-(\mathcal{A})$ is the initial stable ∞-category with a left complete t-structure receiving a map from $\mathcal{A}$, so if our category is left-complete—

N: Yeah sure I'm happy to assume that.

Me: Yeah so in that case we get a map $\mathcal{D}^-(\mathcal{C}^\heartsuit) \to \mathcal{C}$ extending the inclusion $\mathcal{C}^\heartsuit \to \mathcal{C}$.

N: Okay great. So it seems like you're using homological indexing—

Me: Yeah.

N: Yeah so I can't keep up with all the indexing in my head but—

Me: Me neither without having to sit down and think about it.

N: Yeah that's fine. So let's say I now have $k[v]$ where $|v| = 2$, or maybe $|v| = -2$ depending on indexing. Let's just say $|v| = \pm 2$ and you can decide. So let's say I have $k[v]$ and I consider modules over $k[v]$. Can you tell me what the heart is and what the inclusion of the derived category of the heart is?

Me: Sure. So suppose that M is a $k[v]$-module. Then we have that $H^*(M)$ is a graded $H^*(k[v])$-module. If M is in the heart then $H^*(M) = L$ is just a module concentrated in degree 0. Thus if $|v| = 2$ so that the negative degrees of $H^*(k[v])$ don't interfere, then for degree reasons I get that the $H^*(k[v])$-action on L factors through $H^*(k[v])/H^{>0}(k[v]) = k$. So L is just a k-module and the heart is just k-modules. The inclusion of the derived category then just sends a complex of k-modules to a $k[v]$-module where v acts by 0.

N: Great.

N: Fraydoun, do you have any questions you'd like to ask?

The committee laughs.

3. Algebraic Topology ($\sim$30 min)

N: Oh, algebraic topology. Fraydoun this is your opportunity!

T: I guess we'll just start with a classic question... Can you compute the fundamental group of an oriented genus g surface?

I start to write down the gluing diagram of Σ_g as a $4g$-gon and I am going to explain Seifert Van Kampen before getting stopped.

N: You're writing things like a^{-1} and a but also arrows, so are you telling me to glue like... Or how am I supposed to glue?

Me: Well you should glue using the arrows but I guess what I mean by the inverses is that if you traverse the boundary clockwise then that's describing the path you are traversing.

T: (To David) You have to account for the double negations!

They laugh.

T: Can first tell me what you think of when I say genus g surface?

Me: I mean a surface with g holes.

N: What do you mean by g holes?

Me: Like I want $H_1 = \mathbb{Z}^{2g}$...

I draw a picture and a surface with three holes, notably all in a line rather than with a common center which is the picture I should draw if I am referencing the gluing diagram... I also mention one definition as $T^2 \# T^2 \# \cdots \# T^2$.

T: What is going on with the pieces in your picture that you're building the surface with.

I eventually realize that he means that my middle hole is coming from a torus with two discs cut out to form the connect sum.

T: What does that correspond to on your polygon?

I spend a while confused.

N: Maybe we're asking too many things at once. How about you finish the fundamental group computation and then we can return to this.

I compute the fundamental group using Seifert Van Kampen and the gluing diagram.

N: Okay let's go back to you gluing diagram and let's maybe do this for genus 2. Can you draw me the picture and label what the edges and what not from your gluing diagram correspond to on the picture?

I draw the fundamental loops.

N: Okay now let's do it for genus three.

I draw a new picture for genus three, this time realizing that I should have the tori all around a common center.

Me: Okay actually I feel like I should be drawing my picture like this.

N: I agree.

I then try to draw the fundamental loops but I do this using two vertices and too much symmetry than exists in my polygon picture.

N: I think you've drawn to many vertices.

Me: Yeah, they're getting identified in my gluing, maybe all of them? Do you want me to work this out?

N: Yes. I figure you maybe have seen this in your studying but I want to see if you can actually do it.

Me: Yeah. Honestly I just remember the gluing diagram but I haven't done this in years.

N: That's what I figured.

David laughs. I then return to my gluing diagram and realize there is only one vertex. I then draw my fundamental loops coming from my polygon edges on the picture this time all sharing one vertex.

N: Now can you tell me how you're cutting up the polygonal to get the individual tori and where that corresponds to on your picture?

I mistakenly argue that we're just cutting off four edges at a time in a circle with some "extra space" in the middle which I try to point to on the picture.

N: I think you have to much symmetry here. Also, I'm not sure what you mean by this "extra space."

T: How many cuts do you need on your polygon to do this?

I pause a while thinking.

N: Maybe this isn't worth dwelling on—

Me: Oh wait! I should be cutting with rays all originating from a common vertex!

I draw it on my polygon as well as on the three dimensional picture. The confusion has now been resolved.

N: Okay great.

T: There you go.

N: So I guess this is a minor so you wouldn't have higher homotopy groups... Oh you do! In that case let me ask you about Alexander Givental's favorite space. Well, I don't know if it's his favorite space but he's asked about it on all his quals. What can you tell me about $S^1 \vee \mathbb{C}P^\infty$? Say, maybe, its invariants like cohomology and homotopy groups.

I start by mentioning that S^1 is $K(\mathbb{Z},1)$ and that by pullback of the tautological bundle $\mathbb{C}P^\infty$ classifies complex line bundles, and hence is $BU(1) \simeq BS^1$ since we can put a Hermitian metric on every complex line bundle. I then say that the fiber sequence $G \hookrightarrow EG \to BG$ implies that B shifts the homotopy groups up by one, so $BS^1 \simeq BK(\mathbb{Z},1) \simeq K(\mathbb{Z},2)$.

N: What is this K thing you've written?

Me: It's an Eilenberg-MacLane space, so one definition is that it has no higher homotopy groups except for one with the specified group and degree.

N: Where did you learn all of this?

Me: Just life I guess, I don't exactly remember.

N: I don't mean it in a bad way. Your reference is Hatcher; I don't remember this stuff in Hatcher is all.

Me: Oh yeah, I just put that as a reference because I think it covers the stuff I think I know.

I then say that reduced cohomology sends wedges to direct sums. I then state that $\mathbb{C}P^\infty$ has a cell structure implying that $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x]$, $|x| = 2$. They then ask for the ring structure which I say can be done by looking at $\mathbb{C}P^n \subseteq \mathbb{C}P^\infty$ for $n \to \infty$ and use that cup products are dual to the intersection product. I then move on to compute π_1 but indicate that I'm about to use something that doesn't make use of my initial remarks about $K(\mathbb{Z},1)$ and $K(\mathbb{Z},2)$...

G: Wait— is there something on the board already that you could use?

Me: Oh right. I have that $S^1 \vee \mathbb{C}P^\infty \simeq K(\mathbb{Z},1) \vee K(\mathbb{Z},2)$ and π_1 sends wedge sums to free products so I get that π_1 is $\mathbb{Z}$.

I now start to sweat thinking that the higher homotopy groups should be easy given this description but I can't remember how higher homotopy groups behave with wedge sums.

Me: Mmm... Do we have that π sends wedge products to direct sums?

N: No.

I then draw the universal cover of $S^1 \vee \mathbb{C}P^\infty$ showing that it is equivalent to $\bigvee_{\mathbb{Z}} \mathbb{C}P^\infty$.

T: Maybe you can just do it for π_2.

I still cannot figure out how to deal with wedge sums.

N: There's a theorem you could use here, but it's not on your syllabus so maybe you don't know it.

At this point I think maybe he's referring to the infinite Hopf fibration, or something of the sorts. I write that down and start talking about it before realizing that's only going to help me with $\mathbb{C}P^\infty$ which I already understand, not the wedge sum.

T: Let's do something simpler. Can you compute π_2 of $S^2 \vee S^2$ for me? Can you think of a nicer space in which the wedge product lives?

Me: It lives inside the product $S^2 \times S^2$.

T: Good. And how close are they to being the same?

Me: Their quotient is $S^2 \wedge S^2 \simeq S^4$.

T: So what cells do they differ by?

Me: By a four cell. So by cellular approximation they both have the same π_2, so $\pi_2(S^2 \vee S^2) \cong \pi_2(S^2 \times S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$.

T: Good.

I then try to apply that to our situation but going back to the original $S^1 \vee \mathbb{C}P^\infty$.

N: No, no. Don't use that one you'll get extra stuff you don't want.

I return to the $\bigvee_{\mathbb{Z}} \mathbb{C}P^\infty$.

Me: By cellular approximation, we can ignore higher cells and replace $\mathbb{C}P^\infty$ by $\mathbb{C}P^2$, or...

N: No, no you can do better.

Me: Oh sure. I can replace each by $\mathbb{C}P^1 \simeq S^2$. Then by the same argument this has π_2 the same as that of $\prod_{\mathbb{Z}} S^2$. I not sure whether π_2 will commute with infinite products though...

N: Can you do this using colimits?

I remark that $\prod_{\mathbb{Z}} S^2$ is $\operatorname{hocolim}_n \prod_{i=1}^n S^2$ and that S^2 is compact so it commutes with filtered colimits.

Me: Okay so I guess $\pi_2(\prod_{\mathbb{Z}} S^2) = \lim_n \prod_{i=1}^n \mathbb{Z}$...

N: Maybe you mean colimit.

Me: Oh right. Yeah so $\pi_2(\prod_{\mathbb{Z}} S^2) = \operatorname{colim}_n \prod_{i=1}^n \mathbb{Z}$ which are the elements in $\prod_{\mathbb{Z}} S^2$ with finitely many non-zero terms. So it is the infinite direct sum inside of the infinite product.

N: Okay great. The theorem you could have used was Hurewicz.