Loop Spaces and D-Modules

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1 Categories with S^1-action

Definition 1.1. Let $\mathcal{C}$ be an ∞-category, $x \in \mathcal{C}$ and G a topological group (or $\mathbb{E}_1$-space for experts). A G-action on x is a morphism of $\mathbb{E}_1$-algebras (i.e. associative algebras) $G \to \operatorname{Aut}_{\mathcal{C}}(x)$ where $\operatorname{Aut}_{\mathcal{C}}(x)$ is an associative algebra under composition.

Let us unpack what this means. Recall that if X is a space, then $\Omega_x X$ has an associative group structure (up to homotopy) given by concatenation of loops. It turns out that every space with an associative group structure up to homotopy (i.e. $\mathbb{E}_1$-algebra in spaces) arises from this construction—that is to say, if G is an $\mathbb{E}_1$-space, then we may find a *delooping* of G, i.e. a space X such that $G \simeq \Omega_x X$ with the group structure given by concatenation of loops. We denote this delooping by BG.

Moreover, if $f : X \to Y$ is a continuous map, then the induced map $\Omega_x X \to \Omega_{f(x)} Y$ automatically preserves concatenation of loops, i.e. is a map of associative algebras. It turns out that loops and delooping gives a full equivalence of categories (c.f. May recognition theorem), i.e. given two associative algebras H, G we have that

$$\mathbb{E}_1\text{–maps } \ G \to H \iff \text{continuous maps } \ BG \to BH.$$

As a consequence of this, an action of G on $x \in \mathcal{C}$ is the same as a map

$$BG \to B\operatorname{Aut}_{\mathcal{C}}(x).$$

Now, one may view topological spaces as ∞-categories in which all morphisms are invertible, i.e. ∞-groupoids. In this setting, the topological space $B\operatorname{Aut}_{\mathcal{C}}(x)$ is modeled by the one object category $*$ where the endomorphisms of $*$ are given by the topological space $\operatorname{Aut}_{\mathcal{C}}(x)$. In particular, we have a natural inclusion $B\operatorname{Aut}_{\mathcal{C}}(x) \to \mathcal{C}$ where $*$ gets sent to x, i.e. we have a map

$$\begin{aligned} BG &\longrightarrow \mathcal{C} \\ * &\longmapsto x. \end{aligned}$$

It turns out that this, in fact, does not lose any information. Indeed, modeling BG as a one point category, since G is a group BG is a groupoid. Thus a map $BG \to \mathcal{C}$ is equivalently a map $BG \to \mathcal{C}^{\simeq}$ where $\mathcal{C}^{\simeq}$ is given by throwing out all non-invertible morphisms in $\mathcal{C}$. Since BG has only one object, a map $BG \to \mathcal{C}^{\simeq}$ is the same as an object $x \in \mathcal{C}$ and a map $G \simeq \operatorname{End}_{BG}(*) \to \operatorname{End}_{\mathcal{C}^{\simeq}}(x) \simeq \operatorname{Aut}_{\mathcal{C}}(x)$ of $\mathbb{E}_1$-algebras.

Definition 1.2. A G-action on an object $x \in \mathcal{C}$ is a functor $BG \to \mathcal{C}$ which sends $* \in BG$ to x.

Example 1.3. We will care about the case where $G = S^1$. Thus to give an S^1-action on $x \in \mathcal{C}$, we will need to provide a map

$$\mathbb{C}P^{\infty} \simeq BS^1 \to B\operatorname{Aut}_{\mathcal{C}}(x)$$

which is a whole tower of compatible maps $\mathbb{C}P^n \to B\operatorname{Aut}_{\mathcal{C}}(x)$. Passing back to loops, we get an $\mathbb{E}_1$-map $\Omega\mathbb{C}P^n \to \operatorname{Aut}_{\mathcal{C}}(x)$ for all n. Thus an S^1-action is a collection of compatible $\Omega\mathbb{C}P^n$-actions as $n \to \infty$.

Moreover, looking at the action map $S^1 \to \operatorname{Aut}_{\mathcal{C}}(x)$, the loop generating $\pi_1(S^1, 1)$ gives an element of $\pi_1(\operatorname{Aut}_{\mathcal{C}}(x), \mathrm{id}_x)$, i.e. a self-homotopy of the identity.

Definition 1.4. Given a G-action $F : BG \to \mathcal{C}$ on an object x, the

(i) G-(homotopy) fixed point object $x^G \in \mathcal{C}$ is the colimit of F which comes with a map $x^G \to x$

(ii) G-(homotopy) orbits object $x_G \in \mathcal{C}$ is the limit of F which comes with a map $x \to x_G$.

Example 1.5. Suppose that $\mathcal{C}$ is a category with the trivial G-action, i.e. the action map is given by $F : BG \to * \xrightarrow{\mathcal{C}} \mathrm{Cat}_{\infty}$, then $\mathcal{C}^G \simeq \operatorname{Fun}(BG, \mathcal{C})$ is the category of objects with G-action inside $\mathcal{C}$.

2 The Tate construction

Let $\mathcal{C}$ be a k-linear category with S^1-action. The k-linearity of the S^1-action means that $\mathcal{C}^{S^1}$ will be a module over $\mathrm{Mod}_k^{S^1}$ where Mod_k is given the trivial S^1-action. Namely, it will be a module over

$$\mathrm{Mod}_k^{S^1} \simeq \mathrm{QCoh}(\mathrm{pt})^{S^1} \simeq \mathrm{QCoh}(BS^1).$$

Now, $\mathrm{QCoh}(BS^1)$ itself is $\mathcal{O}(BS^1)$-linear and $\mathcal{O}(BS^1) = C^*(BS^1, k) = k[\![u]\!]$, $|u| = 2$, so $\mathcal{C}^{S^1}$ is $k[\![u]\!]$-linear.

Definition 2.1. Let $\mathcal{C}$ be a category with S^1-action. The *Tate construction* is given by

$$\mathcal{C}^{\mathrm{Tate}} \coloneqq \mathcal{C}^{S^1} \otimes_{k[\![u]\!]} k(\!(u)\!).$$

We will also need a modified version of the Tate construction when $\mathcal{C}$ is compactly generated. Indeed, suppose that $\mathcal{C} = \operatorname{Ind} \mathcal{C}^{\omega}$ is compactly generated and that the S^1-action of $\mathcal{C}$ arose from an S^1-action on $\mathcal{C}^{\omega}$. It turns out that taking S^1-invariants does not commute with taking compact objects. Thus we make the following two definitions:

Definition 2.2. Let $\mathcal{C}$ be compactly generated with an S^1-action arising from an S^1-action on $\mathcal{C}^{\omega}$. We define

$$\mathcal{C}^{\omega S^1} \coloneqq \operatorname{Ind}((\mathcal{C}^{\omega})^{S^1}).$$

We may then define

$$\mathcal{C}^{\omega \mathrm{Tate}} \coloneqq \mathcal{C}^{\omega S^1} \otimes_{k[\![u]\!]} k(\!(u)\!).$$

Remark 2.3. It turns out that in geometry, this is too restrictive to use. This is because explicitly referencing compact objects breaks functoriality. Indeed, when $\mathcal{C} = \mathrm{QCoh}(X)$ or $\mathrm{IndCoh}(X)$, then the usual six functors often do not preserve compact objects. However, there is a fix using t-structures (see [Pre15]) and one can define yet a third Tate construction $\mathcal{C}^{t\,\mathrm{Tate}}$ which we will not dwell on. However, this is really the one we should be using.

It turns out that these two constructions are genuinely different. We illustrate this below in the case of a point.

2.1 The Tate construction for a point

Our geometric cases of interest will arise when $\mathcal{C}$ is a category of sheaves on a stack X with S^1-action, so that $\mathcal{C}$ inherits an S^1-action coming from that on X.

For the most basic example, consider $X = \mathrm{pt}$ with the trivial S^1-action. Then we have that $\mathrm{QCoh}(\mathrm{pt})$ also inherits the trivial S^1-action and $\mathrm{QCoh}(\mathrm{pt})^{S^1} \simeq \mathrm{QCoh}(BS^1)$. Let us try to understand the $k[\![u]\!]$-linearity on $\mathrm{QCoh}(BS^1)$ and compute the two Tate constructions. For this, the $\mathcal{O}(BS^1)$-linearity on $\mathrm{QCoh}(BS^1)$ comes from the affinization map

$$\pi : BS^1 \to \mathrm{Aff}(BS^1).$$

This induces an adjoint pair

$$\mathrm{QCoh}(BS^1) \underset{\pi_*}{\overset{\pi^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathrm{QCoh}(\mathrm{Aff}(BS^1)) \simeq \mathcal{O}(BS^1)\text{–mod}.$$

Note that under π^*, $\mathrm{QCoh}(BS^1)$ becomes an algebra (and hence a module) over $\mathcal{O}(BS^1)\text{-mod}$—this is where $\mathrm{QCoh}(BS^1)$ gets its $k[\![u]\!]$-linearity.

Now, the structure sheaf $\mathcal{O}_{BS^1}$ is not compact in $\mathrm{QCoh}(BS^1)$ so $\pi_* = \mathrm{Hom}(\mathcal{O}_{BS^1}, -)$ does not preserve filtered colimits. As such, it is impossible for π_* and π^* to be adjoint equivalences. However, we do have that following fact as a consequence of Koszul duality:

Fact 2.4 (Koszul Duality)**.** Pushforward and pullback along π restrict to adjoint equivalences

$$\mathrm{Coh}(BS^1) \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} \mathrm{Coh}(\mathrm{Aff}(BS^1)).$$

Remark 2.5. In particular, since π^* is $\mathcal{O}(BS^1)$-linear, we get that $\pi_*|_{\mathrm{Coh}(BS^1)}$ is $\mathcal{O}(BS^1)$-linear. This is not obvious a priori (at least to me) since π_* does not need to satisfy a projection formula in this scenario.

Now, $\mathrm{Aff}(BS^1) = \mathrm{Spec}\,\mathcal{O}(BS^1)$ and $\mathcal{O}(BS^1) = k[\![u]\!]$ is smooth, so coherent $k[\![u]\!]$-modules agree with perfect $k[\![u]\!]$-modules. Thus, taking Ind of both sides we have adjoint $k[\![u]\!]$-linear equivalences

$$\mathrm{Ind}(\mathrm{Coh}(BS^1)) \underset{\mathrm{Ind}(\pi_*|_{\mathrm{Coh}(BS^1)})}{\overset{\mathrm{Ind}(\pi^*|_{\mathrm{Coh}(\mathrm{Aff}(BS^1))})}{\rightleftarrows}} \mathrm{Ind}(\mathrm{Coh}(\mathrm{Aff}(BS^1))) \simeq \mathcal{O}(BS^1)\text{-mod}.$$

Let us now see how this impacts the Tate construction for the large category $\mathrm{QCoh}(BS^1)$. To do this, we have a $k[\![u]\!]$-linear inclusion

$$\mathrm{QCoh}(BS^1) \simeq \mathrm{Ind}(\mathrm{Perf}(BS^1)) \subseteq \mathrm{Ind}(\mathrm{Coh}(BS^1))$$

with left inverse given by sending a formal colimit "colim_i"M_i to its actual colimit $\mathrm{colim}_i M_i \in \mathrm{QCoh}(BS^1)$. Now, the composite

$$\mathcal{O}(BS^1)\text{-mod} \simeq \mathrm{Ind}(\mathrm{Coh}(\mathrm{Aff}(BS^1))) \xrightarrow{\mathrm{Ind}(\pi^*|_{\mathrm{Coh}(\mathrm{Aff}(BS^1))})} \mathrm{Ind}(\mathrm{Coh}(BS^1)) \xrightarrow{\mathrm{colim}} \mathrm{QCoh}(BS^1)$$

agree with π^* since π^* preserves colimits. Thus we get the following corollary.

Corollary 2.6. *We have a $k[\![u]\!]$-linear fully faithful inclusion*

$$\mathrm{QCoh}(BS^1) \simeq \mathrm{Ind}(\mathrm{Perf}(BS^1)) \xrightarrow[\mathrm{Ind}(\pi_*|_{\mathrm{Perf}(BS^1)})]{} \mathrm{Ind}(\mathrm{Coh}(\mathrm{Aff}(BS^1))) \xrightarrow[\mathrm{colim}]{\simeq} \mathcal{O}(BS^1)\text{-mod}.$$

This inclusion also has a left inverse given by $\pi^* : \mathcal{O}(BS^1)\text{-mod} \to \mathrm{QCoh}(BS^1)$.

Let us now try to understand the image of $\mathrm{QCoh}(BS^1) \hookrightarrow \mathcal{O}(BS^1)-\mathrm{mod}$ given to us by Corollary 2.6. For this, by descent along $\pi : \mathrm{pt} \to BS^1$, we have that

$$\mathrm{QCoh}(BS^1) \simeq \mathcal{O}(S^1)-\mathrm{comod}$$

where $\mathcal{O}(S^1)$ inherits a coalgebra structure from the multiplication map $m : S^1 \times S^1 \to S^1$. Now, $\mathcal{O}(S^1) \simeq C^*(S^1, k)$ has finite dimensional and bounded cohomology, so it dualizable with dual $C_{-*}(S^1, k)$. Thus we have that

$$\mathrm{QCoh}(BS^1) \simeq C^*(S^1, k)-\mathrm{comod} \simeq C_{-*}(S^1, k)-\mathrm{mod}.$$

where $C_{-*}(S^1, k) \simeq H_{-*}(S^1, k) \simeq k[\lambda]/(\lambda^2)$, $|\lambda| = -1$, is formal. We will leave the quotient implicit from now on so that $\mathrm{QCoh}(BS^1) \simeq k[\lambda]-\mathrm{mod}$.

Under this equivalence, the structure sheaf $\mathcal{O}_{BS^1}$ gets sent to the augmentation module $k = k[\lambda]/(\lambda^2)$ and the pushforward is given by taking $\mathrm{Hom}_{k[\lambda]}(k, -)$ and remembering the $\mathrm{Hom}_{k[\lambda]}(k, k) = k[\![u]\!]$-module structure given by tensoring maps. Now, perfect modules are stably generated by $k[\lambda]$ and we have that

$$\mathrm{Hom}_{k[\lambda]}(k, k[\lambda]) \simeq k \simeq k[\![u]\!]/(u).$$

Hence under the inclusion of Corollary 2.6, Perf BS^1 is contained in the stable subcategory generated by $k[\![u]\!]/(u)$ and thus $\mathrm{QCoh}(BS^1)$ is given by filtered colimits of objects in this subcategory. In particular, we learn that

Corollary 2.7. *The $k[\![u]\!]$-linear inclusion $\mathrm{QCoh}(BS^1) \hookrightarrow k[\![u]\!]-\mathrm{mod}$ of Corollary 2.6 lands inside the subcategory $(k[\![u]\!]-\mathrm{mod})^{u-\mathrm{nil}}$ of locally u-nilpotent modules.*[1]

Thus we learn:

Corollary 2.8. $\mathrm{QCoh}(\mathrm{pt})^{\mathrm{Tate}} \simeq \mathrm{Ind}(\mathrm{Coh}(\mathrm{pt}))^{\mathrm{Tate}} \simeq 0.$

Proof. We have that

$$\mathrm{QCoh}(\mathrm{pt})^{S^1} \simeq \mathrm{QCoh}(BS^1).$$

But there is a $k[\![u]\!]$-linear embedding of $\mathrm{QCoh}(BS^1)$ into $(k[\![u]\!]-\mathrm{mod})^{\mathrm{u-nil}} \subseteq k[\![u]\!]-\mathrm{mod}$. Thus we necessarily have

$$\mathrm{QCoh}(\mathrm{pt})^{\mathrm{Tate}} \simeq \mathrm{QCoh}(BS^1) \otimes_{k[\![u]\!]} k(\!(u)\!) \simeq 0. \qquad \square$$

On the other hand:

Corollary 2.9. $\mathrm{QCoh}(\mathrm{pt})^{\omega\mathrm{Tate}} \simeq \mathrm{Ind}(\mathrm{Coh}(\mathrm{pt}))^{\omega\mathrm{Tate}} \simeq k(\!(u)\!)-\mathrm{mod}.$

[1] Here, a module M being locally u-nilpotent means that every element in $\pi_* M$ is annihilated by some power of u.

2.2 The Tate construction for an affine

Suppose that $X = \operatorname{Spec} R$ is equipped with an S^1-action, so that in particular R becomes an algebra object in $k[\lambda]$-modules. Note that this simply means that R has a $k[\lambda]$-module structure and that the multiplication map is $k[\lambda]$-linear. Since the tensor product on $k[\lambda]-\text{mod}$ is convolution, linearity here imposes a Liebniz rule on the action by λ.

Suppose now that R is coherent as a $k[\lambda]$-module or equivalently that it is coherent as a k-module. Then we have that

$$\text{Coh}(R) \simeq \text{Mod}_R(\text{Coh}(k)).$$

Taking S^1-fixed points results in the category of R-modules with compatible S^1-action, i.e.

$$\text{Coh}(R)^{S^1} \simeq \text{Mod}_R(\text{Coh}(k[\lambda])). \tag{2.2.1}$$

Equivalently, this may be written as

$$\text{Coh}(R)^{S^1} \simeq (R \rtimes k[\lambda])-\text{mod} \simeq R[\Lambda]-\text{mod}$$

where $R[\Lambda]$ is the algebra given by formally adjoining a degree -1 variable Λ and quotienting by the relation that $[\Lambda, r] = \lambda \cdot r$ for every $r \in R$. This may be seen, e.g. in [BZN12], however we opt to write it as (2.2.1) to aid in what follows. Using (2.2.1), we may apply Koszul duality in the case of a point from the previous section to see that

$$\text{Coh}(R)^{S^1} \simeq \text{Mod}_R(\text{Coh}(k[\lambda])) \simeq \text{Mod}_{R^{S^1}}(\text{Perf}(k[\![u]\!])) \simeq \text{Coh}(R^{S^1}).$$

Warning 2.10. I have tried to be somewhat careful with finiteness conditions here, but I have not actually checked they are what they should be.

Now, suppose that $R = (C^\bullet, d)$ is given by a chain complex with internal differential d. To compute R^{S^1}, one may use a resolution of $k = k[\lambda]/(\lambda)$ by free $k[\lambda]$-modules to see that

$$R^{S^1} = (C^\bullet[\![u]\!], d + u\lambda),$$

i.e. λ becomes incorporated into the internal differential. From here one evidently sees the $k[\![u]\!]$-linearity and may invert it.

3 Loop spaces to $\mathcal{D}$-modules

Warning 3.1. Like the previous warning, there is no guarantee of complete technical accuracy in this section. In general, the equivalences we want likely do not restrict immediately down to categories of coherent objects—indeed, as with the usual six functors of IndCoh, coherent objects are not preserved and the fix is to incorporate t-structure magic. A more technically accurate account is given in [Pre15] where t-structures are used to not have to assume all compact objects get preserved.

Recall that we saw an equivalence $\mathcal{L}X \simeq \mathbb{T}_X[-1]$ under which the S^1-action of loop rotation on $\mathcal{L}X$ corresponds to the $B\mathbb{G}_a$-action on the odd tangent bundle given by the de Rham differential. In particular, we have that

$$\mathrm{Coh}(\mathcal{L}X) \simeq \Omega_X^{-\bullet}\text{−mod}(\mathrm{Coh}(X))$$

where $\Omega_X^{-\bullet}$ has a mixed complex structure coming from the de Rham differential. Now, working relative to X over which $\mathcal{L}X \to X$ is affine and $\mathcal{O}_{\mathcal{L}X} = \Omega_X^{-\bullet}$ is a coherent $\mathcal{O}_X$-module, we may locally perform the computations we did in the previous sections to see that

$$\mathrm{Coh}(\mathcal{L}X)^{S^1} \simeq \mathcal{A}\text{−mod}(\mathrm{Coh}(X))$$

where $\mathcal{A} = (\Omega_X^{-\bullet}[\![u]\!], ud_{\mathrm{dR}})$. Thus

$$\mathcal{A} \otimes_{k[\![u]\!]} k(\!(u)\!) \simeq \Omega^{-\bullet}_{X,d_{\mathrm{dR}}} \otimes_k k(\!(u)\!)$$

and so

$$\mathrm{Coh}(\mathcal{L}X)^{S^1} \otimes_{k[\![u]\!]} k(\!(u)\!) \simeq \Omega^{-\bullet}_{X,d_{\mathrm{dR}}}\text{−mod}(\mathrm{Coh}(X)) \otimes_k k[u,u^{-1}].$$

Now, objects of $\Omega^{-\bullet}_{X,d_{\mathrm{dR}}}$−mod$(\mathrm{Coh}(X))$, which are coherent sheaves on X equipped with a $\Omega^{-\bullet}_{X,d_{\mathrm{dR}}}$-module structure are referred to as (coherent) Ω-modules in the literature.

It turns out that we again have a Koszul duality relating Ω-modules to $\mathcal{D}$-modules, referred to as Ω-$\mathcal{D}$-duality in the literature. Morally, this follows from a form of Koszul duality for *Lie algebroids*. In this setting, $\mathcal{D}_X$ is the enveloping algebra of T_X and $\Omega^{-\bullet}_{X,d_{\mathrm{dR}}}$ is the Chevalley–Eilenberg algebra associated to T_X and they are Koszul dual. Koszul duality then gives that

$$\Omega^{-\bullet}_{X,d_{\mathrm{dR}}}\text{−mod}(\mathrm{Coh}(X)) \simeq \mathcal{D}_X\text{−perf}$$

and so we conclude that

$$\mathrm{Ind}(\mathrm{Coh}(\mathcal{L}X)^{S^1}) \otimes_{k[\![u]\!]} k(\!(u)\!) \simeq \mathcal{D}_X\text{−mod} \otimes_k k(\!(u)\!),$$

i.e. we have recovered 2-periodic $\mathcal{D}_X$-modules.

References

[BZN12] David Ben-Zvi and David Nadler. Loop spaces and connections. *Journal of Topology*, 5(2):377–430, 2012.

[Pre15] Anatoly Preygel. Ind-coherent complexes on loop spaces and connections. *Stacks and categories in geometry, topology, and algebra*, 643:289–323, 2015.