# Lie Algebra Cohomology

William Fisher

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**Abstract**

In this expository paper we discuss some basics of Lie algebra cohomology and how it can be used to study the topology of a Lie algebra $\mathfrak{g}$'s integrating Lie group $G$.

## 1 Introduction

Given a Lie algebra $\mathfrak{g}$, Lie's third theorem tells us that there is a unique simply connected Lie group $G$ integrating $\mathfrak{g}$. In light of this, one would expect that any properties of $G$ determined by its smooth isomorphism class should be determinable from $\mathfrak{g}$ alone. In particular, one should be able to compute the cohomology ring $H^*(G)$ of $G$ using $\mathfrak{g}$. Pursuing this route gives a natural notion for the cohomology of a Lie algebra. In this paper, we derive what this notion should be and talk briefly on its extensions and applications.

## 2 From De Rham to Chevalley-Eilenberg

To begin, let us consider a manifold $M$ with (left) $G$-action for some Lie group $G$. We call a differential form $\alpha$ invariant if $g^*\alpha = \alpha$ for all $g \in G$, where by $g^*$ we mean pull-back along the diffeomorphism induced by the action of $g$. As a consequence of pull-back commuting with $d$ and exterior products, it follows that invariant forms of $M$ form a sub-dg-algebra of the de Rham complex $(\Omega^\bullet(M), d)$ which we will denote by $(\Omega^\bullet_L(M), d)$. Provided that $G$ is compact, averaging forms on $M$ over their $G$-translates gives a way to relate these two complexes. In particular,

**Proposition 2.1.** *Assume that $G$ is compact, connected and $M$ is compact. Then the inclusion $i : (\Omega^\bullet_L(M), d) \hookrightarrow (\Omega^\bullet(M), d)$ induces an isomorphism on cohomology.*

*Proof.* Let $T : G \times M \to M$ be the action map. Because $G$ is compact there exists a $G$-invariant Haar measure $\mathrm{d}g$ on $G$ integrating to one. Given a form $\alpha$ on $M$, define

$$\rho(\alpha) = \int_G g^*\alpha \, \mathrm{d}g$$

to be a pointwise average of $\alpha$ over $G$. One checks that this defines a new form $\rho(\alpha)$ on $M$, and since $\mathrm{d}g$ is $G$-invariant we have that $\rho(\alpha) \in \Omega^\bullet_L(M)$. Moreover, if $\alpha \in \Omega^\bullet_L(M)$, then as $\int \mathrm{d}g = 1$ and $g^*\alpha = \alpha$ for all $g$, we have that $\rho(\alpha) = \alpha$.

Finally, since pull-back commutes with $d$, $\rho$ gives a chain map $\rho : (\Omega^\bullet(M), d) \to (\Omega^\bullet_L(M), d)$ such that $\rho \circ i = \mathrm{id}$. On the other hand, under the assumption that $G$ is connected and $M$ is compact, one may show that $i \circ \rho$ is chain homotopic to $\mathrm{id}$ (see [1, Chapter IV]). Taking induced maps on cohomology, we obtain the desired result. □

At this point, we specialize to the case of $M = G$ with $G$ acting on itself by left-multiplication. Since this action is transitive, invariant forms on $G$ are determined by their value at the identity and in fact a stronger statement can be made:

**Proposition 2.2.** *The map*

$$\begin{aligned} \Omega_L^k(G) &\longrightarrow \textstyle\bigwedge^k \mathfrak{g}^* \\ \alpha &\longmapsto \alpha_e \end{aligned}$$

*sending left-invariant forms to their value at the identity is a bijection.*

Carrying over the exterior derivative via this identification, we obtain an isomorphism $(\Omega_L^\bullet(G), d) \cong (\bigwedge^\bullet \mathfrak{g}^*, D)$ of dg-algebras. Provided we can identify the differential $D$ under this correspondence, we will then have succeeded in our goal of computing the cohomology ring of $G$ from $\mathfrak{g}$ alone, at least when $G$ is compact, connected.

**Proposition 2.3.** *The differential $D$ given by the exterior derivative under the isomorphism $\Omega_L^\bullet(G) \cong \bigwedge^\bullet \mathfrak{g}^*$ is the dual of the Lie bracket, i.e. $D = [\cdot,\cdot]^* : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$.*

*Proof.* Since $(\bigwedge^\bullet \mathfrak{g}^*, D)$ is a dg-algebra, it suffices to determine $D : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$. For this, given $v \in \mathfrak{g}^*$ let $\alpha_v$ denote the corresponding left-invariant 1-form and given $w \in \mathfrak{g}$ let $l_w$ denote the corresponding left-invariant vector field.

Given $v \in \mathfrak{g}^*$ and $w \in \mathfrak{g}$, we have that $\iota_{l_w}(\alpha_v)$ is also $G$-invariant, hence constant. Thus by Cartan's magic formula we have that

$$\begin{aligned} \iota_{l_w} \mathrm{d}\alpha_v &= \mathcal{L}_{l_w}\alpha_v - \mathrm{d}\iota_{l_w}\alpha_v \\ &= \mathcal{L}_{l_w}\alpha_v. \end{aligned} \tag{2.0.1}$$

But $\Phi^t = R_{\exp(tw)}$, where $R_g$ denotes right-multiplication by $g$, is the flow of $l_w$. Thus

$$\begin{aligned} (\mathcal{L}_{l_w}\alpha_v)_e(u) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (R^*_{\exp(tw)}\alpha_v)_e(u) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} v(C^*_{\exp(tw)}u) \\ &= v([w,u]). \end{aligned}$$

It follows from (2.0.1) that

$$\begin{aligned} (\iota_{l_w}\mathrm{d}\alpha_v)_e(u) &= (\mathrm{d}\alpha_v)_e(w,u) \\ &= v([w,u]) \end{aligned}$$

which is what we wanted to prove. □

Having identifying the Lie algebra side of the picture, the resulting dg-algebra $(\bigwedge^\bullet \mathfrak{g}^*, D)$ deserves a name.

**Definition 2.4.** Given a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ over $k$, the *Chevalley-Eilenberg algebra* of $\mathfrak{g}$, denoted $\mathrm{CE}(\mathfrak{g})$, is given by $\mathrm{CE}(\mathfrak{g}) = (\bigwedge_k^\bullet \mathfrak{g}^*, d)$ where $d$ is the dual of the Lie bracket.

*Remark* 2.5. This construction is functorial, and the fact that the differential $d$ in $\mathrm{CE}(\mathfrak{g})$ squares to zero is precisely the condition that $[\cdot,\cdot]$ satisfies the Jacobi identity. As a consequence, we get a fully faithful functor $\mathrm{CE} : \mathrm{LieAlg}_k \to \mathrm{dgAlg}_k^{\mathrm{op}}$ whose essential image is precisely those algebras whose underlying complex is the exterior algebra of some vector space. This gives an alternative definition of a Lie algebra. Moreover, because this functor is fully faithful, $\mathrm{CE}(\mathfrak{g})$ depends only on the isomorphism class of $\mathfrak{g}$.

We can then conclude with the following theorem which is an immediate corollary of the work above.

**Theorem 2.6.** *If $G$ is a compact, connected Lie group, then the cohomology ring $H^*(G,\mathbb{R})$ depends only on its associated Lie algebra and we have that $H^*(G,\mathbb{R}) \cong H^*(\mathrm{CE}(\mathfrak{g}))$.*

*Remark* 2.7. Since $\mathfrak{g}$ determines $G$ only when $G$ is simply-connected, the condition that $G$ be connected should not come as a surprise. This in mind, the only constraint we have truly imposed is that $G$ be compact and in fact we have shown something non-trivial: If $G$ is compact, connected then the cohomology ring of $G$ is determined by $\mathfrak{g}$, despite $\mathfrak{g}$ not-necessarily determining $G$ itself.

# 3 Extensions and Applications

More generally, given a $\mathfrak{g}$-module $M$, one may consider $G$-invariant forms on $G$ valued in $M$. Continuing a similar line of argumentation to that above, one arrives at the notion of Lie algebra cohomology with coefficients in $M$ and the above consideration becomes the special case of $M = \mathbb{R}$, the trivial $\mathfrak{g}$-module.

More algebraically, through some work, one can see that the Chevalley-Eilenberg complex constructed above is simply giving us a resolution to compute the right derived functors of the left-exact functor $\mathrm{Hom}_{U\mathfrak{g}}(\mathbb{R}, -)$ where $\mathbb{R}$ is the trivial module. Thus one may also make the following definition.

**Definition 3.1.** Given a Lie algebra $\mathfrak{g}$ over $R$ and a $\mathfrak{g}$-module $M$, the cohomology of $\mathfrak{g}$ with coefficients in $M$ is defined to be $H^*(\mathfrak{g}; M) = \mathrm{Ext}^*_{U\mathfrak{g}}(R, M)$.

This approach lends itself to manipulations using the full power of homological algebra, though from its definition is less geometrically motivated. One famous result that this line of research leads to is Weyl's theorem which we state but whose proof is outside the scope of this paper.

**Theorem 3.2** (Weyl)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic zero. Then every finite dimensional $\mathfrak{g}$-module is semisimple.*

# References

[1] Werner Greub, Stephen Halperin, and Ray Vanstone. *Connections, Curvature, and Cohomology Vol II: Lie Groups, Principal Bundles, and Characteristic Classes*. Academic press, 1972.