An Introduction to Abelian and non-Abelian Hodge Theory

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1 Introduction

The non-abelian Hodge correspondence relates flat bundles over a compact Kähler manifold to holomorphic objects known as Higgs bundles. The goal of this paper is to give an exposition of non-abelian Hodge correspondence as it generalizes the classical Hodge decomposition. One may view non-abelian Hodge theory as one of many results sitting in a web of categorifications of classical statements in topology, differential geometry and algebraic geometry.

To the see the start of this story, recall a classical theorem of de Rham which tells us that for a smooth manifold X,

$$H^k(X, \mathbb{C}) \cong H^k_{\mathrm{dR}}(X) \otimes \mathbb{C}.$$

This relates the cohomology of the local system $\underline{\mathbb{C}}$ on X to the cohomology of the complex

$$\mathcal{A}^0_{\mathbb{C}}(X) \xrightarrow{d} \mathcal{A}^1_{\mathbb{C}}(X) \xrightarrow{d} \mathcal{A}^2_{\mathbb{C}}(X) \xrightarrow{d} \cdots$$

associated to the trivial line bundle $\mathcal{A}^0_{\mathbb{C}}(X)$ with flat connection d. However, this is just one instance of a more general correspondence. There exists a well-known equivalence of categories

$$\Big\{\mathbb{C}\text{-local systems over } X\Big\} \xrightarrow{\sim} \left\{ \substack{\mathbb{C}\text{-vector bundles with} \\ \text{flat connection over } X} \right\} \tag{1.0.1}$$

which preserves the natural notion of cohomology in both categories. Indeed, given a flat bundle (E, ∇) we may consider the hypercohomology of the complex

$$\mathcal{A}^0(E) \xrightarrow{\nabla} \mathcal{A}^1(E) \xrightarrow{\nabla} \mathcal{A}^2(E) \xrightarrow{\nabla} \cdots . \tag{1.0.2}$$

These sheaves are fine, hence acyclic, and by standard ODE theory (1.0.2) gives a resolution of $\ker(\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E))$. In particular, if one defines the cohomology of (E, ∇) to be the hypercohomology of (1.0.2) and the cohomology of the local system $\mathcal{H}$ to be $H^\bullet(X, \mathcal{H})$, then the equivalence (1.0.1) preserves these notions. Thus, as $\underline{\mathbb{C}}$ corresponds to $(\mathcal{A}^0_{\mathbb{C}}(X), d)$ under (1.0.1), (1.0.1) may viewed as a categorification of the de Rham isomorphism.

Alongside the de Rham isomorphism, we also have the classical Hodge decomposition. If X

is a compact Kähler manifold, we may write

$$H^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^q(X,\Omega_X^p) = \mathbb{H}^k(X, \bigoplus_i \Omega_X^i[i]).$$

This relates the cohomology of the trivial local system $\underline{\mathbb{C}}$ to that of a complex

$$\mathcal{O}_X \xrightarrow{0} \Omega_X^1 \xrightarrow{0} \Omega_X^2 \xrightarrow{0} \cdots .$$

This complex is reminiscent to the de Rham complex, except now we've replaced the trivial complex line bundle with the trivial holomorphic line bundle $\mathcal{O}_X$, and the flat connection d with 0. Thus we may be hopeful that the classical Hodge decomposition, just as the de Rham isomorphism, arises as a special case of a broader equivalence of objects. In particular, one may hope for a chain of equivalences

$$\Big\{\mathbb{C}\text{-local systems over } X\Big\} \xrightarrow{\sim} \left\{\begin{matrix}\mathbb{C}\text{-vector bundles with}\\ \text{flat connection over } X\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{holomorphic vector bundles +}\\ \text{some extra data over } X\end{matrix}\right\} \tag{1.0.3}$$

which is "cohomology preserving" and recovers the de Rham isomorphism and Hodge decomposition when applied to the simplest case of trivial line bundles and local system:

$$\underline{\mathbb{C}} \longleftrightarrow (\mathcal{A}^0_{\mathbb{C}}(X), d) \longleftrightarrow (\mathcal{O}_X, 0).$$

The first work suggesting such an equivalence is due to Narasimhan and Seshadri [10]. For a compact Riemann surface X, they established an equivalence between unitary representations of $\pi_1(X)$ and stable holomorphic vector bundles over X. In a series of generalizations due to Donaldson [5, 6] and Uhlenbeck and Yau [14], this equivalence was extended to Kähler manifolds of arbitrary dimension. Passing through the Riemann-Hilbert correspondence, which gives an equivalence between flat connections on X and $\mathbb{C}$-representations of $\pi_1(X)$, these results represent a special case of the desired equivalence (1.0.3).

To achieve a more general correspondence, more than just holomorphic vector bundles $\mathcal{E}$ are required. The extra data one needs is that of a Higgs field: a holomorphic $\mathcal{E}nd(\mathcal{E})$-valued 1-form $\phi \in H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega_X^1)$ which commutes with itself, i.e. $\phi \wedge \phi = 0$. Such a pair $(\mathcal{E}, \phi)$ is referred to as a Higgs bundle, and comes with a natural associated complex

$$\mathcal{E} \xrightarrow{\phi} \mathcal{E} \otimes \Omega_X^1 \xrightarrow{\phi} \mathcal{E} \otimes \Omega_X^2 \xrightarrow{\phi} \cdots$$

which is a complex by the condition that $\phi \wedge \phi = 0$. The hypercohomology of this gives a natural notion of cohomology for $(\mathcal{E}, \phi)$. The non-Abelian Hodge theorem gives us a cohomology preserving correspondence between flat bundles on X and Higgs bundles over X, both subject to certain stability conditions.

Under this correspondence, $(\mathcal{A}^0_{\mathbb{C}}(X), d)$ corresponds to $(\mathcal{O}_X, 0)$ and taking cohomology re-

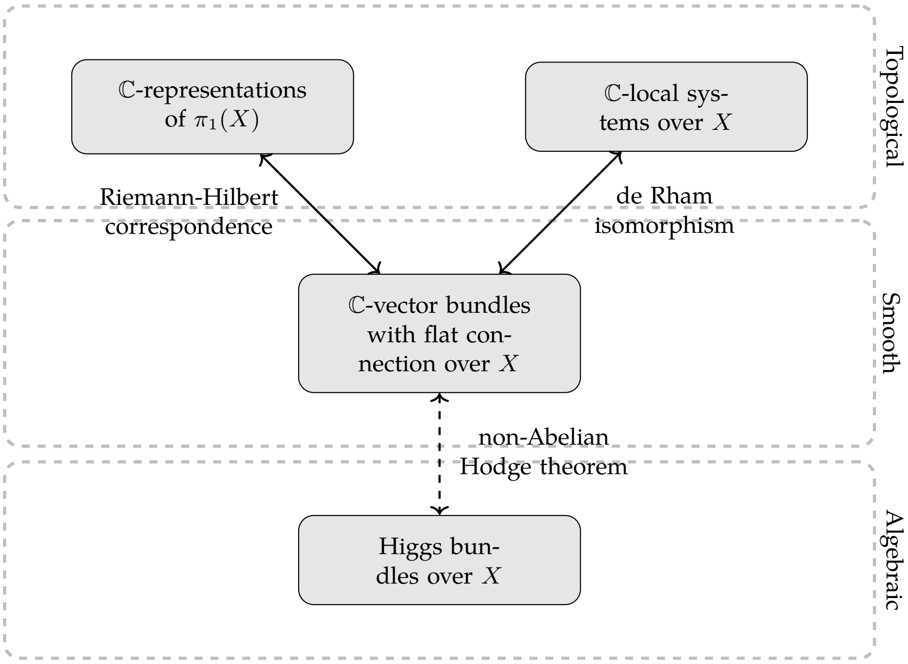

Figure 1: Diagram of equivalent categories for X compact Kähler. Dashed arrow denotes equivalence between full subcategories.

covers the classical Hodge decomposition. In this way the non-Abelian Hodge theorem gives a categorification of the Hodge decomposition. This correspondence is the cumulative result of a long series of generalizations, with the final form being primarily due to Simpson [12] and Corlette [3]. There exist many good introductory resources on this topic [2,7,13]. The goal of this paper is to present abelian and non-abelian Hodge theory together in a way that illuminates how the latter is a generalization of the former.

Outline: The outline of this paper is as follows.
Preliminaries (Sections 2-3): Section 2 covers various categorical preliminaries, particularly hypercohomology and can be skipped by readers already comfortable with this concept. Section 3 covers differentials and connections. These notions are introduced in atypical generality (following [7]) to encompass many related ideas and alleviate definitional burden throughout the rest of the paper.
Abelian Hodge theory (Section 4): Section 4 covers abelian Hodge theory. This refers to the classical statement of Hodge decomposition. The content of this section is thus very similar to a traditional proof of the Hodge decomposition, but is approached in sufficient generality so as to later give us preservation of cohomology in the non-abelian Hodge correspondence.
Non-Abelian Hodge correspondence (Section 5): Section 5 gives the statements of the non-abelian Hodge theorem with a sketch of its proof. Combined with the previous section on abelian Hodge

theory we complete the categorification promised in the introduction.

2 Categorical preliminaries

2.1 Hypercohomology

For this section, we will assume that the reader has already met sheaf cohomology. A good reference for sheaf cohomology and the contents of this section are [15, §4] and [15, §8.1] respectively.

Recall that one defines sheaf cohomology

$$H^i(X, \mathcal{F}) := R^i\Gamma(\mathcal{F})$$

as the right-derived functor of the global sections functor Γ. To compute this, one takes an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ and sets $R^i\Gamma(\mathcal{F}) = H^i(\Gamma(\mathcal{I}^\bullet))$. The goal of hypercohomology is to generalize right-derived functor $R^i\Gamma$ to take values on *left-bounded* complexes $\mathcal{M}^\bullet$ of sheaves, i.e. complexes of sheaves with $\mathcal{M}^n = 0$ for n sufficiently negative, rather than just sheaves $\mathcal{F}$ which we may view as a left-bounded complex supported in degree 0.

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories in which $\mathcal{A}$ has *enough injectives*[1] and let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor.

Definition 2.1. Let $A^\bullet$ and $B^\bullet$ be complexes in an abelian category. A *quasi-isomorphism* $f : A^\bullet \to B^\bullet$ is a chain map such that $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism for all i.

Provided that $\mathcal{A}$ has enough injectives, any left-bounded complex $M^\bullet$ is quasi-isomorphic to a complex of injective objects of $\mathcal{A}$, and such a choice is essentially unique.

Lemma 2.2. *Let $M^\bullet$ be a left-bounded complex in $\mathcal{A}$. Then there exists a complex $I^\bullet$ with each I^n injective and a quasi-isomorphism $f : M^\bullet \to I^\bullet$. Moreover, given another such choice $g : M^\bullet \to J^\bullet$, there exists a morphism $\phi : I^\bullet \to J^\bullet$ which is unique up to homotopy with $\phi \circ f = g$.*

Proof. See [15, Proposition 8.4] and [15, Lemma 8.7]. □

This allows us to extend the definition of R^iF to left-bounded complexes by setting

$$R^iF(M^\bullet) := H^i(F(I^\bullet))$$

where $I^\bullet$ is any complex of injective objects quasi-isomorphic to $M^\bullet$, as guaranteed by Lemma 2.2. Of course, this depends on our choice of $I^\bullet$. However, if one is also willing to remember the choice of quasi-isomorphism $f : M^\bullet \to I^\bullet$, then we have

Proposition 2.3. *$R^iF(M^\bullet)$ is well-defined up to canonical isomorphism.*

[1] This is a technical condition that applies to all categories we will consider. This means that for all $A \in \mathrm{Ob}(\mathcal{A})$ we may find a monomorphism $A \to I$ with I an injective object of $\mathcal{A}$.

Proof. Let $g : M^\bullet \to J^\bullet$ be another choice of quasi-isomorphism with a complex of injectives. Then Lemma 2.2 gives a morphism $\phi : I^\bullet \to J^\bullet$ with $\phi \circ f = g$ and similarly a $\psi : J^\bullet \to I^\bullet$ with $\psi \circ g = f$. By uniqueness, we must have $\phi \circ \psi \simeq \mathrm{id}_{J^\bullet}$ and $\psi \circ \phi \simeq \mathrm{id}_{I^\bullet}$ and thus ϕ and ψ are homotopy equivalences. Hence $F(\phi)$ is also a homotopy equivalence, and thus a quasi-isomorphism.

Moreover, as ϕ is unique up to homotopy, so is $F(\phi)$. Thus $H^i(F(\phi))$ is independent of ϕ and gives a canonical isomorphism $H^i(F(\phi)) : H^i(F(I^\bullet)) \to H^i(F(J^\bullet))$. □

In the special case of $F = \Gamma$ and $\mathcal{A}$ the category of sheaves over X with values in $\mathcal{B}$, where $\mathcal{B}$ has enough injectives, one typically writes

$$R^i\Gamma(\mathcal{M}^\bullet) =: \mathbb{H}^i(X, \mathcal{M}^\bullet)$$

and refers to $\mathbb{H}^i(X, \mathcal{M}^\bullet)$ as the *hypercohomology* of $\mathcal{M}^\bullet$.

Example 2.4. Let $\mathcal{F}$ be a sheaf and $\mathcal{F} \to \mathcal{K}^\bullet$ an injective resolution. Then, by definition, one has $H^i(X, \mathcal{F}) = \mathbb{H}^i(X, \mathcal{K}^\bullet)$. Moreover, if one views $\mathcal{F}$ as a complex supported in degree zero, then one also has $H^i(X, \mathcal{F}) = \mathbb{H}^i(X, \mathcal{F})$. It is in this sense that hypercohomology generalizes regular sheaf cohomology.

Just as in sheaf cohomology, $R^iF(-)$ is functorial. That is, given $\alpha : M^\bullet \to N^\bullet$, we get a morphism $R^iF(\alpha) : R^iF(M^\bullet) \to R^iF(N^\bullet)$ and when each N^k is acyclic for the functor F, $R^iF(\alpha)$ becomes an isomorphism. This allows one to compute hypercohomology using acyclic complexes, which we will use later.

We will also need a slightly technical proposition which allows computing hypercohomology from double complexes, which we state now.

Proposition 2.5. *Let $(A^\bullet, D)$ be the simple complex associated to the double complex $(A^{\bullet,\bullet}, D_1, D_2)$. Suppose also that we have a morphism $i : M^\bullet \to (A^{\bullet,0}, D_1)$ with each i^p injective such that each $M^p \xrightarrow{i^p} (A^{p,\bullet}, D_2)$ is a resolution of M^p. Then the composite*

$$M^\bullet \to (A^{\bullet,0}, D_1) \to (A^\bullet, D)$$

is a quasi-isomorphism.

Proof. See [15, Lemma 8.5]. □

3 Differentials and connections

This section introduces differentials and connections in the manner of [7, §2]. Let $(\mathfrak{X}, \mathcal{O})$ be a locally ringed space with $\mathcal{O}$ a sheaf of $\mathbb{C}$-algebras.

3.1 Differentials

Definition 3.1. Let $\mathcal{K}$ be a locally free sheaf of $\mathcal{O}$-modules over $\mathfrak{X}$. A differential d on $\mathcal{K}$ is a collection of $\mathbb{C}$-linear maps $d^{(n)} : \bigwedge_{\mathcal{O}}^{n} \mathcal{K} \to \bigwedge_{\mathcal{O}}^{n+1} \mathcal{K}$ such that

(i) for all $v \in \bigwedge_{\mathcal{O}}^{n_1} \mathcal{K}$, $w \in \bigwedge_{\mathcal{O}}^{n_2} \mathcal{K}$

$$d^{(n_1+n_2)}(v \wedge w) = d^{(n_1)}(v) \wedge w + (-1)^{n_2} v \wedge d^{(n_2)}(w)$$

(ii) for all n, $d^{(n+1)} \circ d^{(n)} = 0$

That is, $d = \bigoplus_n d^{(n)} : \bigwedge_{\mathcal{O}}^{\bullet} \mathcal{K} \to \bigwedge_{\mathcal{O}}^{\bullet} \mathcal{K}$ is a degree 1 $\mathbb{C}$-linear derivation which squares to zero.

Remark 3.2. As slight abuse of notation, when we write $v \in \bigwedge_{\mathcal{O}}^{n} \mathcal{K}$ or $v \in \mathcal{F}$ for any other sheaf $\mathcal{F}$, we mean that $v \in \mathcal{F}(U)$ is a section of $\mathcal{F}$ over some open $U \subseteq X$.

Every differential d on $\mathcal{K}$ then gives an associated complex

$$\mathcal{O} \xrightarrow{d} \mathcal{K} \xrightarrow{d} \bigwedge\nolimits_{\mathcal{O}}^{2} \mathcal{K} \xrightarrow{d} \bigwedge\nolimits_{\mathcal{O}}^{3} \mathcal{K} \xrightarrow{d} \cdots . \qquad (3.1.1)$$

Example 3.3. Given a smooth manifold M, we may take $(\mathfrak{X}, \mathcal{O}) = (M, \mathcal{A}^0_{\mathbb{C}}(M))$. Then the complex extension of the exterior derivative d gives a differential on $\mathcal{A}^1_{\mathbb{C}}(M)$, the sheaf of complex 1-forms on M. The associated complex to d is the *(complex) de Rham complex* of M, the hypercohomology of which computes the complexified de Rham cohomology $H^{\bullet}_{\text{dR}}(M, \mathbb{C}) = H^{\bullet}_{\text{dR}}(M) \otimes \mathbb{C}$.

Example 3.4. Given a complex manifold X, we may take $(\mathfrak{X}, \mathcal{O}) = (X, \mathcal{A}^0_{\mathbb{C}}(X))$. Then the operator $\bar{\partial}$ gives a differential on $\mathcal{A}^{0,1}(X)$ whose associated complex

$$\mathcal{A}^{0,0}(X) \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1}(X) \xrightarrow{\bar{\partial}} \mathcal{A}^{0,2}(X) \xrightarrow{\bar{\partial}} \cdots$$

is called the *Dolbeault complex* of X, and has hypercohomology which computes the Dolbeault cohomology of X.

Example 3.5. Let X be a complex manifold and $\mathcal{O}_X$ the sheaf of holomorphic functions on X. More generally, we may consider

$$\Omega^p_X = \ker(\bar{\partial} : \mathcal{A}^{p,0}(X) \to \mathcal{A}^{p+1,0}(X)),$$

the sheaf of holomorphic p-forms. Then the exterior derivative $d = \partial + \bar{\partial}$ sends holomorphic p-forms to holomorphic $(p+1)$-forms, so it gives a differential on Ω^1_X with respect to the pair $(\mathfrak{X}, \mathcal{O}) = (X, \mathcal{O}_X)$. To emphasize that the underlying locally ringed space is $(X, \mathcal{O}_X)$ rather than $(X, \mathcal{A}^0_{\mathbb{C}}(X))$ or $(X, \mathcal{A}^{1,0}(X))$, we will try to write d_h for the restriction of d to holomorphic p-forms. Note that $d_h = d = \partial$ all agree on holomorphic p-forms. The associated complex

$$\mathcal{O}_X \xrightarrow{d_h} \Omega^1_X \xrightarrow{d_h} \Omega^2_X \xrightarrow{d_h} \Omega^3_X \xrightarrow{d_h} \cdots$$

is called the *holomorphic de Rham complex* of X, and its hypercohomology also computes the complex de Rham cohomology of X.

3.2 λ-D-connections

Given a differential d on $\mathcal{K}$, we may then define connections relative to d. Just as in the classical case of d the exterior derivative, connections on a vector bundle $\mathcal{V}$ should take sections of $\mathcal{V}$ to $\mathcal{V}$-valued forms, and in this setting $\mathcal{K}$ replaces our sheaf of 1-forms.

Definition 3.6. Let $\mathcal{V}$ be a locally free sheaf of $\mathcal{O}$-modules over X, and d a differential on $\mathcal{K}$. Then a *λ-d-connection*, $\lambda \in \mathbb{C}$, on $\mathcal{V}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}} \mathcal{K}$ satisfying the λ-twisted Liebniz rule

$$\nabla(fs) = \lambda s \otimes d(f) + f\nabla(s)$$

for all $f \in \mathcal{O}$ and $s \in \mathcal{V}$.

Given a λ-d-connection ∇, we may extend it to a collection of maps $\nabla^{(n)} : \mathcal{V} \otimes_{\mathcal{O}} \bigwedge_{\mathcal{O}}^{n} \mathcal{K} \to \mathcal{V} \otimes_{\mathcal{O}} \bigwedge_{\mathcal{O}}^{n+1} \mathcal{K}$ by setting

$$\nabla^{(n)}(s \otimes w) = \lambda s \otimes d(w) + \nabla(s) \wedge w$$

for $s \in \mathcal{V}$ and $w \in \bigwedge_{\mathcal{O}}^{n} \mathcal{K}$.

One then has

$$\begin{aligned}(\nabla^{(1)} \circ \nabla^{(0)})(fs) &= \nabla^{(1)}(\lambda s \otimes df + f\nabla s)\\ &= \lambda^2 s \otimes d^2 f + \lambda \nabla s \wedge df - \lambda \nabla s \wedge df + f(\nabla^{(1)} \circ \nabla^{(0)})(s)\\ &= f(\nabla^{(1)} \circ \nabla^{(0)})(s).\end{aligned}$$

Thus $\nabla^{(1)} \circ \nabla^{(0)}$ is $\mathcal{O}$-linear and under the identifications

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}}(\mathcal{V}, \mathcal{V} \otimes_{\mathcal{O}} \textstyle\bigwedge_{\mathcal{O}}^{2}\mathcal{K}) &= \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{E}nd(\mathcal{V}) \otimes_{\mathcal{O}} \textstyle\bigwedge_{\mathcal{O}}^{2}\mathcal{K})\\ &= \Gamma(X, \mathcal{E}nd(\mathcal{V}) \otimes_{\mathcal{O}} \textstyle\bigwedge_{\mathcal{O}}^{2}\mathcal{K})\end{aligned}$$

we may view $\nabla^{(1)} \circ \nabla^{(0)} \in \Gamma(X, \mathcal{E}nd(\mathcal{V}) \otimes_{\mathcal{O}} \bigwedge_{\mathcal{O}}^{2}\mathcal{K})$.

Definition 3.7. The *curvature* of ∇ is defined to be $\nabla^{(1)} \circ \nabla^{(0)} \in \Gamma(X, \mathcal{E}nd(\mathcal{V}) \otimes_{\mathcal{O}} \bigwedge_{\mathcal{O}}^{2}\mathcal{K})$, which we may abbreviate as $\nabla \circ \nabla$. We say that ∇ is *flat* or a *flat connection* if $\nabla \circ \nabla = 0$.

Given a flat connection ∇, we also have an associated complex

$$\mathcal{V} \xrightarrow{\nabla} \mathcal{V} \otimes_{\mathcal{O}} \mathcal{K} \xrightarrow{\nabla} \mathcal{V} \otimes_{\mathcal{O}} \textstyle\bigwedge_{\mathcal{O}}^{2} \mathcal{K} \xrightarrow{\nabla} \mathcal{V} \otimes_{\mathcal{O}} \textstyle\bigwedge_{\mathcal{O}}^{3} \mathcal{K} \xrightarrow{\nabla} \cdots \qquad (3.2.1)$$

The hypercohomology of this complex provides a natural notion for what the cohomology of the flat bundle $(\mathcal{V}, \nabla)$ ought to mean. Note also that any differential d on $\mathcal{K}$ is itself a flat 1-d-connection on $\mathcal{K}$, and the associated complex (3.1.1) is the same as (3.2.1).

Example 3.8 (Holomorphic vector bundles). Algebraically, one may define a holomorphic vector bundle over a complex manifold X as a locally free sheaf of $\mathcal{O}_X$-modules, where $\mathcal{O}_X$ is the sheaf of holomorphic functions on X. Translating this into differential geometric terms, a holomorphic vector bundle over X is a smooth complex vector bundle over X with a distinguished cover of trivializations whose transition functions are holomorphic.

Given such a vector bundle $E \to X$ with sheaf of smooth sections $\mathcal{E}$ we may define an operator $\bar{\partial}_E : \mathcal{E} \to \mathcal{E} \otimes \mathcal{A}^{0,1}(X)$ locally under a distinguished trivialization

$$E|_U \cong U \times \mathbb{C}^n$$

by

$$\bar{\partial}_E(f_1, \dots, f_n) = (\bar{\partial} f_1, \dots, \bar{\partial} f_n). \tag{3.2.2}$$

Because E has holomorphic transition functions, this definition is well defined, and one checks that

$$\bar{\partial}_E(fs) = s \otimes \bar{\partial} f + f \bar{\partial}_E s,$$

so $\bar{\partial}_E$ is a 1-$\bar{\partial}$-connection on E. Moreover, using the local form (3.2.2), one sees that $\bar{\partial}_E^2 = 0$, so $\bar{\partial}_E$ is a flat connection. The holomorphic sections of E are then precisely those sections annihilated by $\bar{\partial}_E$.

Conversely, given a smooth complex vector bundle E with flat 1-$\bar{\partial}$-connection $\bar{\partial}_E$, it is a theorem of Koszul-Malgrange [9] that we may find trivializations of E with holomorphic transition functions such that the construction above reproduces $\bar{\partial}_E$. In light of this, holomorphic vector bundles over X may be viewed in three equivalent ways

(i) A locally free sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$

(ii) A smooth complex vector bundle $E \to X$ with a distinguished collection of trivializations covering X with holomorphic transition functions

(iii) A smooth complex vector bundle $E \to X$ with flat 1-$\bar{\partial}$-connection $\bar{\partial}_E$.

We will freely switch between these equivalent definitions throughout the essay.

Now, vector bundles with λ-d-connections may be turned into a category $\mathbf{Conn}_\lambda$ and given a λ-d-connection ∇, $\mu\nabla$ becomes a $\mu\lambda$-d-connection. This $\mathbb{C}^\times$-action shows that for all $\lambda \neq 0$, $\mathbf{Conn}_\lambda$ is equivalent to $\mathbf{Conn}_1$. Thus the case $\lambda = 0$ immediately distinguishes itself and in the holomorphic setting, vector bundles with 0-d_h-connections are what we call *Higgs bundles*.

Example 3.9 (Higgs bundles). Let X be a complex manifold. Then a holomorphic vector bundle $\mathcal{E}$ with 0-d_h-connection ϕ is referred to as a Higgs bundle, written together as $(\mathcal{E}, \phi)$. Here ϕ is referred to as the *Higgs field*.

In the $\lambda = 0$ case, the 0-d_h-connection ϕ is $\mathcal{O}_X$-linear, so we may also view $\phi \in \Gamma(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega_X^1)$. In this case, we often write the flatness condition as $\phi \wedge \phi = 0$ and it may be viewed as a self-commutivity condition. Indeed, if one locally writes, after trivializing $\mathcal{E}$ and taking complex coordinates $z_1, \ldots, z_n$ for X,

$$\phi = \sum_i A_i \mathrm{d}z_i$$

then $\phi \wedge \phi = 0$ says exactly that the A_i are pairwise commuting.

We can also give a description of Higgs bundles in the smooth setting. In this case we require a triple $(E, \bar{\partial}_E, \theta)$ where $\bar{\partial}_E$ is a flat 1-$\bar{\partial}$-connection and θ is a flat 0-∂-connection such that

$$\bar{\partial}_E \theta + \theta \bar{\partial}_E = 0. \tag{3.2.3}$$

In this case, by the discussion in Example 3.8, $(E, \bar{\partial}_E)$ gives us our holomorphic bundle $\mathcal{E}$. Moreover, the anti-commutivity constraint (3.2.3) tells us that θ descends to a flat 0-d_h-connection on $\mathcal{E}$, giving the Higgs field. In this case we will sometimes also write (E, D) for the Higgs bundle where $D = \bar{\partial}_E + \theta$.

4 Abelian Hodge theory

4.1 A motivating limiting case

We begin this section with an observation that naturally leads one to conjecture the classical Hodge decomposition.

Let X be a complex manifold. Then we have a natural family of differentials $t \cdot d_h$, $t \in \mathbb{C}$, on the pair $(\mathcal{O}_X, \Omega_X^1)$. This gives a family of flat t-d_h-connections $\nabla_t = t \cdot d_h$ on the trivial holomorphic bundle $\mathcal{O}_X$. One might then ask, how does the cohomology of these flat bundles vary in this family? Intuitively, one might expect behavior to split into the two cases of $t = 0$ and $t \in \mathbb{C}^*$. This is indeed the case as the following propositions show.

Proposition 4.1. *For $t \in \mathbb{C}^*$, we have that $\mathbb{H}^k(X, (\Omega_X^\bullet, t \cdot d_h)) = H^k(X, \mathbb{C})$.*

Proof. We have a quasi-isomorphism $(\Omega_X^\bullet, t \cdot d_h) \to (\Omega_X^\bullet, d_h)$ given by

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{t \cdot d_h} & \Omega_X^1 & \xrightarrow{t \cdot d_h} & \Omega_X^2 & \xrightarrow{t \cdot d_h} & \Omega_X^3 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \times 1} & & \downarrow{\scriptstyle \times t^{-1}} & & \downarrow{\scriptstyle \times t^{-2}} & & \downarrow{\scriptstyle \times t^{-3}} & & \\
0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{d_h} & \Omega_X^1 & \xrightarrow{d_h} & \Omega_X^2 & \xrightarrow{d_h} & \Omega_X^3 & \longrightarrow & \cdots
\end{array}.$$

Thus it suffices to show this for the case $t = 1$, i.e. show that the hypercohomology of the holomorphic de Rham complex computes the cohomology of the de Rham complex.

For this, we have an inclusion $(\Omega_X^\bullet, d_h) \hookrightarrow (\mathcal{A}_{\mathbb{C}}^{\bullet;0}, \partial)$. Moreover, for each fixed p, we have that $\Omega_X^p \hookrightarrow (\mathcal{A}_{\mathbb{C}}^{p;\bullet}, \bar{\partial})$ is a resolution of Ω_X^p. Thus by Proposition 2.5 we have that $(\Omega_X^\bullet, d_h)$ is quasi-isomorphic to the simple complex $(\mathcal{A}_{\mathbb{C}}^\bullet(X), d)$ associated to the double complex $(\mathcal{A}_{\mathbb{C}}^{\bullet;\bullet}, \partial, \bar{\partial})$ and the result follows. $\square$

On the other hand, when $t = 0$, we have:

Proposition 4.2. *We have that* $\mathbb{H}^k(X, (\Omega_X^\bullet, 0)) = \bigoplus_{p+q=k} H^q(X, \Omega_X^p)$.

Proof. Indeed, as complexes, we have that

$$(\Omega_X^\bullet, 0) = \bigoplus_p \Omega_X^p[p]$$

where $\Omega_X^p[p]$ denotes the complex with single non-zero term Ω_X^p supported in degree p. Thus

$$\begin{aligned}\mathbb{H}^k(X, (\Omega_X^\bullet, 0)) &\cong \bigoplus_p \mathbb{H}^k(X, \Omega_X^p[p]) \\ &= \bigoplus_p H^{k-p}(X, \Omega_X^p) \\ &= \bigoplus_{p+q=k} H^q(X, \Omega_X^p)\end{aligned}$$

as required. □

This of course begs the question: Given a holomorphic bundle $\mathcal{E}$ over X with a family ∇_t of flat t-d_h-connections for t in a neighborhood of $0 \in \mathbb{C}$, when is the cohomology preserved in the limit $t \to 0$? The above shows that for the family $(\mathcal{O}_X, t \cdot d_h)$, this question is equivalent to a canonical decomposition

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^q(X, \Omega_X^p).$$

It turns out that in answering this question, it is beneficial to fix an underlying complex smooth vector bundle E, and let the holomorphic structure $\mathcal{E}$ on E vary in t as well. Abelian Hodge theory will then seek to answer this question in the case of Kähler manifolds for families $(\mathcal{E}_t, \nabla_t)$ on E of holomorphic structures with flat t-d_h-connections when our holomorphic structures $\mathcal{E}_t$ and connections ∇_t deform in t in a manner governed by the Kähler metric h.

Later, we will see that the non-abelian Hodge correspondence identifies exactly which pairs of holomorphic bundles $(\mathcal{E}_1, \nabla_1)$ with flat 1-d_h-connections and Higgs bundles $(\mathcal{E}_0, \nabla_0)$ can be related by a family $(\mathcal{E}_t, \nabla_t)$ for which the results of abelian Hodge theory applies. Combining these two, the non-abelian Hodge correspondence will give us an equivalence between full subcategories of the category of flat bundles and the category of Higgs bundles, and the abelian Hodge theorem will show this is cohomology preserving.

4.2 The Hodge Theorem for elliptic complexes

4.2.1 L^2-inner product and Hodge $*$-operator

Many of the technical tools that go into proving the abelian Hodge theorem involve functional analysis, and to create a setting where these techniques apply, we need to topologize the sections of vector bundles over our manifold using an inner product.

Let M be a smooth manifold with volume form vol. Given a Hermitian vector bundle (E, h) over M, we may turn the global sections $A^0(E)$ an inner product space as follows.

Definition 4.3. The *L^2-inner product* on $A^0(E)$ is given by

$$\langle \alpha, \beta \rangle_{L^2} = \int_M \langle \alpha, \beta \rangle_h \text{vol} \tag{4.2.1}$$

where at least one of α, β is compactly supported so that the integral is defined. When M is compact, this turns $A^0(E)$ into an inner product space.

When multiple Hermitian bundles are present, we will sometimes write $\langle \ , \ \rangle_E$ for the L^2-inner product on E to emphasize the which inner product is being used.

Remark 4.4. When M is compact, the assumption that one of α, β be compactly supported is vacuous. While the results we care about will assume M is compact, the proofs will reference the L^2-inner product on open subsets $U \subseteq M$, hence the need for the more general definition.

While this definition alone covers all the cases that we need, it is often the case that we want to consider E-valued k-forms, i.e. global sections of $\Omega^k_{M,\mathbb{R}} \otimes_{\mathbb{R}} E$. While it is certainly possible to independently equip these with Hermitian metrics for each k, the more common scenario is to consider the case when (M, g) is a Riemannian manifold and E is Hermitian, in which case $\Omega^k_{M,\mathbb{R}} \otimes_{\mathbb{R}} E$ comes with an *induced* Hermitian metric.

To illustrate this scenario, assume that (M, g) is an oriented Riemannian manifold. In this case, M has a canonical volume form vol_g given locally by

$$\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n$$

where $\omega_1, \ldots, \omega_n$ is an oriented orthonormal coframe. Moreover, the metric g on TM induces Riemannian metrics on $\Omega^k_{M,\mathbb{R}}$ for all k. Indeed, the metric on $\Omega^1_{M,\mathbb{R}} = T^*M$ is such that for each $x \in M$ and orthonormal basis $v_1, \ldots, v_n$ of T_xM, the dual basis $v_1^*, \ldots, v_n^* \in T_x^*M$ is an orthonormal basis for T_x^*M. Then, for $\Omega^k_{M,\mathbb{R}} = \bigwedge^k_{\mathbb{R}} T^*M$ we assert that in each fibre, if $w_1, \ldots, w_n \in T_x^*M$ is an orthonormal basis, then $\{w_{i_1} \wedge \cdots \wedge w_{i_k} : i_1 < \cdots < i_k\}$ is an orthonormal basis for $(\Omega^k_{M,\mathbb{R}})_x$. Now, if (E, h) is a Hermitian vector bundle over M, then $\Omega^k_{M,\mathbb{R}} \otimes_{\mathbb{R}} E$ has a natural Hermitian metric given by *tensorial extension*, i.e. in each fibre we define

$$\langle \omega_1 \otimes e_1, \omega_2 \otimes e_2 \rangle = \langle \omega_1, \omega_2 \rangle_g \langle e_1, e_2 \rangle_h.$$

For this reason, when (M, g) is an oriented Riemannian manifold and (E, h) is a Hermitian vector bundle over M, the L^2-inner product on $A^k(E)$ will, unless otherwise stated, mean that we take M to have volume form vol_g and $\Omega^k_{M,\mathbb{R}} \otimes_{\mathbb{R}} E$ to have the induced Hermitian metric described above.

Moreover, in this special case, one is able to formulate $\langle \ , \ \rangle_{L^2}$ in terms of the Hodge $*$-

operator. Let $\dim_{\mathbb{R}} M = n$. For every $x \in M$ we have a non-degenerate pairing

$$\begin{array}{ccccc} \Omega^k_{M,\mathbb{R},x} \times \Omega^{n-k}_{M,\mathbb{R},x} & \longrightarrow & \Omega^n_{M,\mathbb{R},x} & \xrightarrow{\cong} & \mathbb{R} \\ (\omega_1, \omega_2) & \longmapsto & \omega_1 \wedge \omega_2 & \longmapsto & \langle \omega_1 \wedge \omega_2, \mathrm{vol}_g \rangle_g \end{array}$$

which induces an isomorphism

$$\Omega^{n-k}_{M,\mathbb{R},x} \cong \mathrm{Hom}(\Omega^k_{M,\mathbb{R},x}, \mathbb{R}).$$

On the other hand, the extension of g to $\Omega^k_{M,\mathbb{R}}$ also induces an isomorphism

$$\Omega^k_{M,\mathbb{R},x} \cong \mathrm{Hom}(\Omega^k_{M,\mathbb{R},x}, \mathbb{R}).$$

Composing these two isomorphisms gives the *Hodge $*$-operator*

$$*_x : \Omega^k_{M,\mathbb{R},x} \xrightarrow{\cong} \Omega^{n-k}_{M,\mathbb{R},x}$$

at the point x. This definition varies smoothly in x and gives a bundle isomorphism

$$* : \Omega^k_{M,\mathbb{R}} \xrightarrow{\cong} \Omega^{n-k}_{M,\mathbb{R}}. \tag{4.2.2}$$

To extend the Hodge $*$-operator to work with E-valued forms, note that the Hermitian metric h on E induces an isomorphism

$$\overline{E} \xrightarrow{\cong} E^\vee.$$

Tensoring this with (4.2.2) gives the Hodge $*$-operator

$$*_E : \Omega^k_{X,\mathbb{R}} \otimes_{\mathbb{R}} \overline{E} \xrightarrow{\cong} \Omega^{n-k}_{M,\mathbb{R}} \otimes_{\mathbb{R}} E^\vee$$

for which we also write $*_E$ for the induced map on sections, i.e. on $\overline{E}$-valued forms.

The definition of $*_E$ gives that for $\alpha, \beta \in A^k(E)$,

$$(\alpha \wedge *_E \overline{\beta})(x) = \langle \alpha, \beta \rangle_{\Omega^k_{M,\mathbb{R}} \otimes_{\mathbb{R}} E}(x) \mathrm{vol}_g(x)$$

where on the left-hand side $\wedge$ denotes exterior product on the form components and contraction of the E and $E^\vee$ components. Thus we find that

$$\langle \alpha, \beta \rangle_{L^2} = \int_M \alpha \wedge *_E \overline{\beta}. \tag{4.2.3}$$

4.2.2 Elliptic operators and orthogonal decomposition

In this subsection we introduce some of the tools from functional analysis that go into the proof of the Hodge decomposition. Seeing as this is distinct from the rest of the essay, we will take the essential results of this section as given.

The setting for this section again will be the real smooth category, as none of the results here require any complex structures on our manifold.

Definition 4.5. Let M be a smooth manifold with complex vector bundles E, F. A $\mathbb{C}$-linear morphism $L : \mathcal{A}^0(E) \to \mathcal{A}^0(F)$ is called a *differential operator* of order k if for every $x \in M$ we may find an open neighborhood $x \in U$ on which M has coordinates $x_1, \dots, x_n$ and $E|_U$, $F|_U$ are trivial and under the identifications

$$E|_U \cong U \times \mathbb{C}^p, F|_U \cong U \times \mathbb{C}^q, \tag{4.2.4}$$

we have that

$$\begin{aligned} L(U) : \mathcal{A}^0_{\mathbb{C}}(M)^p &\longrightarrow \mathcal{A}^0_{\mathbb{C}}(M)^q \\ (\alpha_1, \dots, \alpha_p) &\longmapsto (\beta_1, \dots, \beta_q) \end{aligned}$$

is given by

$$\beta_i = \sum_{j,|I| \le k} L^I_{ij} \frac{\partial \alpha_j}{\partial x^I} \tag{4.2.5}$$

with each $L^I_{ij} \in \mathcal{A}^0_{\mathbb{C}}(M)(U)$ and some $L^I_{ij} \neq 0$ for $|I| = k$.

In other words, L is locally given by a linear partial differential operator of order k. One may check that if L is a differential operator of order k, then the above holds on every open coordinate chart U of M on which E, F are trivial. Note that here when we say that L is $\mathbb{C}$-linear, we are using the complex vector bundle structures of E and F to give $\mathcal{A}^0(E)$ and $\mathcal{A}^0(F)$ the structure of $\underline{\mathbb{C}}$-modules by fibre-wise operations on sections.

For the sake of computation, we will want a coordinate-free description of differential operators, which we quote.

Fact 4.6. *A $\mathbb{C}$-linear morphism $L : \mathcal{A}^0(E) \to \mathcal{A}^0(F)$ is*

(i) *a differential operator of order* 0 *if and only if for every $f \in A^0_{\mathbb{C}}(M)$ we have that the commutator of L and multiplication by f is zero, i.e.* $[L, f] = 0$

(ii) *a differential operator of order k if and only if for every $f \in A^0_{\mathbb{C}}(M)$ we have that the commutator $[L, f]$ is a differential operator of order $k - 1$*

Example 4.7. Differential operators of order k on the trivial bundle $\mathbb{C} \times \mathbb{R}^n \to \mathbb{R}^n$ are precisely linear partial differential operators of order k.

Example 4.8. The exterior derivative $d : \mathcal{A}^k_{\mathbb{C}}(M) \to \mathcal{A}^{k+1}_{\mathbb{C}}(M)$ between sections of the complex vector bundles $\Omega^k_{M,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and $\Omega^{k+1}_{M,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is a differential operator of order 1. Indeed, in the case

$k = 0$, on a coordinate chart U of M with coordinates $x_1, \dots, x_n$ we have that with respect to the trivializing sections 1 and $\mathrm{d}x_1, \dots, \mathrm{d}x_n$, d is given by

$$f \longmapsto \Big(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\Big).$$

Alternatively, using Fact 4.6, one may check that

$$[d, f]\omega = \mathrm{d}(f\omega) - f\mathrm{d}\omega = \mathrm{d}f \wedge \omega$$

so $[d, f] = \mathrm{d}f \wedge (-)$ which is a differential operator of order 0. Hence d is a differential operator of order 1.

Now, although the defining property of a differential operator is independent of the choice of open cover on which it is checked, the exact L^I_{ij} occurring in (4.2.5) do not transform in any particularly nice way as we change between trivializing opens. Indeed, changing coordinates for $\partial/\partial x^I$ will introduce difficult to control linear combinations of $\partial/\partial x^J$ for $|J| < |I|$. This motivates us to look only at the top degree terms and for every open U as in Definition 4.5 consider the matrix $(L_{ij})_{ij}$ where

$$L_{ij} \coloneqq \sum_{|I|=k} L^I_{ij} \frac{\partial}{\partial x^I} \in \Gamma(U, S^k TM)$$

is a local section over U of the k-th symmetric power of TM given by viewing

$$\frac{\partial}{\partial x^I} = \frac{\partial}{\partial x^{i_1}} \cdot \dots \cdot \frac{\partial}{\partial x^{i_k}} \in \Gamma(U, S^k TM)$$

where $I = (i_1 < \dots < i_k)$.

Assembling the L_{ij} into a matrix, we get a section of $\mathrm{Hom}_{\mathbb{C}}(\underline{\mathbb{C}}^p, \underline{\mathbb{C}}^q) \otimes_{\mathbb{R}} S^k TM$ over U, which after reapplying the isomorphisms (4.2.4) gives us

$$(L_{ij})_{ij} \in \Gamma(U, \mathrm{Hom}_{\mathbb{C}}(E, F) \otimes_{\mathbb{R}} S^k TM).$$

A quick exercise shows that these local definitions transform as required to patch together and give a global section

$$\tilde{\sigma}_k(L) \in \Gamma(M, \mathrm{Hom}_{\mathbb{C}}(E, F) \otimes_{\mathbb{R}} S^k TM). \tag{4.2.6}$$

Now, let $\pi : T^*M \to M$ be the cotangent bundle. Then we get a section

$$\pi^* \tilde{\sigma}_k(L) \in \Gamma(M, \mathrm{Hom}_{\mathbb{C}}(\pi^* E, \pi^* F) \otimes_{\mathbb{R}} S^k \pi^* TM) \tag{4.2.7}$$

by pulling back. There is a canonical bundle morphism

$$\begin{aligned} \pi^*TM &\longrightarrow \underline{\mathbb{R}} \\ (\xi, v) &\longmapsto (\pi(\xi), \xi(v)) \end{aligned}$$

given by contraction of the base-point $\xi \in T^*_{\pi(\xi)}M$ with the vector $v \in T_{\pi(\xi)}M$. Applying the k-th symmetric power S^k and considering the product map $S^k\underline{\mathbb{R}} \to \underline{\mathbb{R}}$ we get a composite

$$\begin{aligned} S^k\pi^*TM \longrightarrow S^k&\underline{\mathbb{R}} \longrightarrow \underline{\mathbb{R}} \\ (x, \xi_1 \cdot \dots \cdot \xi_k) &\longmapsto (x, \xi_1 \cdots \xi_k). \end{aligned}$$

Applying this to the factor of $S^k\pi^*TM$ in (4.2.7) we finally get a section

$$\psi_L \in \Gamma(M, \mathrm{Hom}_{\mathbb{C}}(\pi^*E, \pi^*F) \otimes_{\mathbb{R}} \underline{\mathbb{R}}) = \Gamma(M, \mathrm{Hom}_{\mathbb{C}}(\pi^*E, \pi^*F)).$$

Definition 4.9. The section $\sigma_k(L) \coloneqq i^k\psi_L$ of $\mathrm{Hom}_{\mathbb{C}}(\pi^*E, \pi^*F)$ called the *symbol* of L.

Remark 4.10. Sometimes (4.2.6) is taken as the definition of the symbol of L. Moreover, some definitions exclude the factor of i^k, however this normalization will be useful later in making Proposition 4.14(i) hold without additional signs. More concretely, the symbol $\sigma_k(L)$ of L on an open U as in Definition 4.5 is given by the matrix $(A_{ij})_{ij}$ where

$$A_{ij}(x, \sum_\ell \xi_\ell \cdot \mathrm{d}x_\ell) = i^k \sum_{|I|=k} L^I_{ij}(x)\xi_I$$

and $\xi_I = \prod_r \xi_{i_r}$ if $I = (i_1 < \cdots < i_r)$.

Again there is a coordinate-free description of the symbol of L which will sometimes be useful for computations, which we quote.

Fact 4.11. *Given $L : \mathcal{A}^0(E) \to \mathcal{A}^0(F)$ a differential operator of order k, then for $x \in M$, $s \in \Gamma(M, E)$ and $f \in \mathcal{C}^\infty(M, \mathbb{R})$ with $f(x) = 0$ we have that*

$$\sigma_k(L)(x, (\mathrm{d}f)_x)(s(x)) = \frac{i^k}{k!} L(f^k \cdot s)(x).$$

Example 4.12. Returning to Example 4.8, we use Fact 4.11 to compute the symbol of d. We have that for a k-form ω,

$$\sigma_1(d)(x, (\mathrm{d}f)_x)(\omega_x) = i \cdot d(f \cdot \omega)_x = i(\mathrm{d}f \wedge \omega)_x$$

using that $f(x) = 0$. Thus $\sigma_1(d)(x, \xi)(\omega) = i\xi \wedge \omega$.

The symbol of a differential operator will be useful later as an object encoding just enough information about L to get strong yet general results by constraining its behaviour.

Differential operators are useful as they have formal adjoints with respect to the L^2-inner product.

Proposition 4.13. *Let (M, vol) be a smooth manifold with volume form and E, F Hermitian vector bundles over M so that the L^2-inner product is defined. Let $L : \mathcal{A}^0(E) \to \mathcal{A}^0(F)$ be a differential operator of order k. Then there exists a unique formal L^2-adjoint $L^* : \mathcal{A}^0(F) \to \mathcal{A}^0(E)$ in the sense that, for every $U \subseteq M$ open and $\alpha \in \mathcal{A}^0(E)(U)$, $\beta \in \mathcal{A}^0(F)(U)$ with α compactly supported, we have*

$$\langle L(U)(\alpha), \beta \rangle_{L^2} = \langle \alpha, L^*(U)(\beta) \rangle_{L^2}.$$

Moreover, L^* is a differential operator of order k.

Proof. We sketch a proof, see [16, Chapter IV] for more details.

First, if an adjoint exists then it is unique. Indeed, let Q_1, Q_2 be two adjoints to L. Then, over any open $U \subseteq M$, we have that for sections $\alpha \in \mathcal{A}^0(E)(U)$, $\beta \in \mathcal{A}^0(F)(U)$ that

$$\begin{aligned} \langle \alpha, (Q_1(U) - Q_2(U))\beta \rangle &= \langle \alpha, Q_1(U)\beta \rangle - \langle \alpha, Q_2(U)\beta \rangle \\ &= \langle L(U)\alpha, \beta \rangle - \langle L(U)\alpha, \beta \rangle \\ &= 0. \end{aligned}$$

Since this holds for all α, β with α compactly supported, using bump functions we find that $Q_1(U) = Q_2(U)$.

Given uniqueness, we then reduce existence to a local statement by gluing. For this, we may assume that E, F are trivial with the standard metric and $M \cong \mathbb{R}^n$, and that L takes the form in (4.2.5). Moreover, by Moser's theorem, transforming M by a diffeomorphism, we may assume that vol is the Euclidean volume form. Then by linearity and looking at each component, we reduce to the case of finding an adjoint to $f \frac{\partial}{\partial x^I}$ for $f : \mathbb{R}^n \to \mathbb{C}$ some smooth function, but this has adjoint $(-1)^{|I|} \frac{\partial}{\partial x^I}(f \cdot (-))$ by Stokes' theorem. □

Symbols of differential operators are compatible with adjoints and composition in the following sense:

Proposition 4.14. *With the notation of Proposition 4.13 we have that*

(i) $\sigma(L^*)(x, \xi) = \sigma(L)(x, \xi)^*$ where the adjoint here is with respect to the Hermitian inner products on E_x and F_x

(ii) if $M : \mathcal{A}^0(F) \to \mathcal{A}^0(G)$ is another differential operator, then $\sigma(M \circ L) = \sigma(M) \circ \sigma(L)$.

Proof. Since $\sigma(L|_U) = \sigma(L)|_U$, both these statements may be checked locally. Now, locally, L is of the form

$$\sum_{|I| \leq k} (L^I_{ij})_{ij} \frac{\partial}{\partial x^I}, \tag{4.2.8}$$

so claim (ii) follows from the coordinate definition of $\sigma(L)$ by an application of the chain rule.

Next, since we have that $(L_1 \circ L_2)^* = L_2^* \circ L_1^*$ and $(L_1 + L_2)^* = L_1^* + L_2^*$, using (4.2.8) it suffices to prove (i) for matrix multiplication, i.e. differential operators of order 0, and $\frac{\partial}{\partial x^I}$. For differential operators of order 0, (ii) is clear. For the second case, using the same reductions as in Proposition 4.13, we have that the adjoint of $\frac{\partial}{\partial x^I}$ is $(-1)^I \frac{\partial}{\partial x^I}$ which have symbols

$$\sigma\Big(\frac{\partial}{\partial x^I}\Big)(x,\xi) = i^{|I|}\xi_I$$

and

$$\sigma\Big((-1)^{|I|}\frac{\partial}{\partial x^I}\Big)(x,\xi) = (-i)^{|I|}\xi_I.$$

Since ξ_I is real, the result follows. □

Given that adjoints exist, a wishful conjecture motivated by finite dimensional linear algebra is that $\ker L(M)$ and $\operatorname{im} L^*(M)$, which are L^2-orthogonal subspaces, give a decomposition of $A^0(E)$. In finite dimensions, this fact is proven by dimension counting, but here $A^0(E)$ is infinite dimensional. In general, it turns out that this conjecture is false, but when our differential operator L is *elliptic* we do get such a decomposition, which is the crux of this section.

Definition 4.15. A differential operator L is *elliptic* if for every $(x, \xi) \in T^*M$, $\xi \neq 0$, the symbol $\sigma_k(L)(x,\xi) \in \operatorname{Hom}_{\mathbb{C}}(E_x, F_x)$ is injective.

We can then finally state the fundamental result on elliptic differential operators.

Theorem 4.16. *Let (M, vol) be smooth manifold with volume form and E, F Hermitian vector bundles of the same rank over M, and suppose that $L : \mathcal{A}^0(E) \to \mathcal{A}^0(F)$ is an elliptic differential operator. Then*

(i) $\ker L(M) \subseteq A^0(E)$ is finite-dimensional

(ii) $\operatorname{im} L(M) \subseteq A^0(F)$ is closed (in the induced L^2-norm topology) and of finite codimension

(iii) there is an L^2-orthogonal direct sum decomposition $A^0(E) = \ker L(M) \oplus \operatorname{im} L^*(M)$.

4.2.3 Elliptic complexes and the Hodge Theorem

Having gone through the preliminaries of defining the L^2-inner product and discussing the necessary tools from differential and functional analysis, we can begin to apply the results to Hodge theory. Our first goal is to find a source of elliptic operators to which the theory of Section 4.2.2 applies. One such source are elliptic complexes first introduced by Atiyah and Bott [1], which extends elliptic operators, being operators with injective symbols, to collections of operators whose symbols fit into an exact sequence.

Let (M, vol) again be a smooth manifold with volume form.

Definition 4.17. Let $E_1, \ldots, E_N$ be Hermitian vector bundles over M. A sequence of differential operators

$$\mathcal{A}^0(E_0) \xrightarrow{L_0} \mathcal{A}^0(E_1) \xrightarrow{L_1} \mathcal{A}^0(E_2) \xrightarrow{L_2} \cdots \xrightarrow{L_{N-1}} \mathcal{A}^0(E_N) \qquad (4.2.9)$$

is called an *elliptic complex* if $L_{j+1} \circ L_j = 0$ for all j and the associated sequence of symbols

$$0 \longrightarrow \pi^* E_0 \xrightarrow{\sigma(L_0)} \pi^* E_1 \xrightarrow{\sigma(L_1)} \cdots \xrightarrow{\sigma(L_{N-1})} \pi^* E_N \longrightarrow 0$$

is exact away from the zero section, i.e. exact when restricted to each fibre $(x, \xi) \in T^*M$, $\xi \neq 0$.

Given an elliptic complex $(E_\bullet, L_\bullet)$ we define the *cohomology* of $E_\bullet$ to be

$$H^k(E_\bullet, L_\bullet) \coloneqq H^k(\Gamma(\mathcal{A}^0(E_\bullet), L_\bullet)) = \frac{\ker(L_k(M) : A^0(E_k) \to A^0(E_{k+1}))}{\operatorname{im}(L_{k+1}(M) : A^0(E_{k-1}) \to A^0(E_k))},$$

i.e. the cohomology of the global sections of the complex (4.2.9).

Remark 4.18. In our case, the sheaves in (4.2.9) are fine so we have that $H^k(E_\bullet, L_\bullet) = \mathbb{H}^k(M, (\mathcal{A}^0(E_\bullet), L_\bullet))$.

Example 4.19 (de Rham complex). We saw in Example 4.8 that the exterior derivative $d : \mathcal{A}^k_{\mathbb{C}}(M) \to \mathcal{A}^{k+1}_{\mathbb{C}}(M)$ is a differential operator, and these operators fit into a complex

$$\mathcal{A}^0_{\mathbb{C}}(M) \xrightarrow{d} \mathcal{A}^1_{\mathbb{C}}(M) \xrightarrow{d} \mathcal{A}^2_{\mathbb{C}}(M) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^n_{\mathbb{C}}(M)$$

with $n = \dim_{\mathbb{R}} M$ called the (*complex*) *de Rham complex*. By Example 4.12, this has associated symbol complex at the point (x, ξ) given by

$$0 \longrightarrow \Omega^0_{M,\mathbb{C},x} \xrightarrow{i\xi\wedge(-)} \Omega^1_{M,\mathbb{C},x} \xrightarrow{i\xi\wedge(-)} \cdots \xrightarrow{i\xi\wedge(-)} \Omega^n_{M,\mathbb{C},x} \longrightarrow 0.$$

which by basic linear algebra is exact when $\xi \neq 0$. Thus the de Rham complex is elliptic.

As their name suggests, elliptic complexes are a convenient source of elliptic operators.

Definition 4.20. Let $(E_\bullet, L_\bullet)$ be an elliptic complex. The associated *Laplacians* are given by $\Delta_j = L_{j-1}L^*_{j-1} + L^*_j L_j : \mathcal{A}^0(E_j) \to \mathcal{A}^0(E_j)$.

One immediately sees that the Δ_j are self-adjoint differential operators, but they also are elliptic. Indeed, Proposition 4.14 shows that

$$\sigma(\Delta_j) = \sigma(L_{j-1}) \circ \sigma(L_{j-1})^* + \sigma(L_j)^* \circ \sigma(L_j).$$

Evaluating at points $(x, \xi) \in T^*M$, $\xi \neq 0$, then reduces injectivity of $\sigma(\Delta_j)(x, \xi)$ to following linear algebra statement.

Lemma 4.21. *Given an exact sequence*

$$0 \longrightarrow V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{N-1}} V_N \longrightarrow 0 \tag{4.2.10}$$

of finite-dimensional $\mathbb{C}$*-inner product spaces, each* $f_{j-1} \circ f^*_{j-1} + f^*_j \circ f_j$ *is injective.*

Proof. Write $F_j = f_{j-1} \circ f_{j-1}^* + f_j^* \circ f_j$. Suppose $F_j x = 0$. Then we have that

$$\begin{aligned} 0 &= \langle x, F_j x \rangle \\ &= \langle f_{j-1}^* x, f_{j-1}^* x \rangle + \langle f_j x, f_j x \rangle. \end{aligned}$$

Thus $f_{j-1}^* x = 0$ and $f_j x = 0$. By exactness of (4.2.10) we can write $x = f_{j-1} y$ and so

$$0 = \langle y, f_{j-1}^* x \rangle = \langle x, x \rangle.$$

Thus $x = 0$. □

Now, let M be assumed to be compact so that the L^2-inner product turns each $A^0(E_j)$ into an inner product space. Then we have

Proposition 4.22. *Let $\alpha \in A^0(E_j)$. Then $\Delta_j \alpha = 0$ if and only if $L_j \alpha = 0$ and $L_{j-1}^* \alpha = 0$.*

Proof. One direction is obvious. For the other direction, we have that

$$0 = \langle \alpha, \Delta_j \alpha \rangle_{E_j} = \langle L_{j-1}^* \alpha, L_{j-1}^* \alpha \rangle_{E_{j-1}} + \langle L_j \alpha, L_j \alpha \rangle_{E_{j+1}},$$

so $\Delta_j \alpha = 0$ implies $L_j \alpha = 0$ and $L_{j-1}^* \alpha = 0$. □

We give sections $\alpha \in A^0(E_j)$ satisfying $\Delta_j \alpha = 0$ a special name.

Definition 4.23. Let $\alpha \in A^0(E_j)$. If $\Delta_j \alpha = 0$ we say that α is *harmonic* and we denote by $\mathcal{H}(E_j)$ the space of harmonic sections in $A^0(E_j)$.

Applying the fundamental Theorem 4.16 of elliptic operators, we arrive at

Theorem 4.24 (Hodge Theorem). *Let $(E_\bullet, L_\bullet)$ be an elliptic complex. Then for every k we have that*

(*i*) *$H^k(E_\bullet, L_\bullet)$ is finite-dimensional*

(*ii*) *there is an L^2-orthogonal direct sum decomposition $A^0(E_k) = \mathcal{H}(E_j) \oplus \operatorname{im} L_{k-1}(M) \oplus \operatorname{im} L_k^*(M)$*

(*iii*) *the natural map $\mathcal{H}(E_k) \to H^k(E_\bullet, L_\bullet)$ is an isomorphism.*

Proof. We have that Δ_k is an elliptic operator, so by Theorem 4.16 we get that $\ker \Delta_k$ is finite-dimensional and

$$A^0(E_k) = \ker \Delta_k \oplus \operatorname{im} \Delta_k^* = \ker \Delta_k \oplus \operatorname{im} \Delta_k$$

seeing as Δ_k is self-adjoint. Now, $\operatorname{im} \Delta_k \subseteq \operatorname{im} L_{k-1}(M) \oplus \operatorname{im} L_k^*(M)$ but on the other hand if $\gamma \in \ker \Delta_k$, then

$$\begin{aligned} \langle L_{k-1}\beta_1 + L_k^* \beta_2, \gamma \rangle_{E_k} &= \langle \beta_1, L_{k-1}^* \gamma \rangle_{E_{k-1}} + \langle \beta_2, L_k \gamma \rangle_{E_{k+1}} \\ &= 0 \end{aligned}$$

for all β_1, β_2 by Proposition 4.22. Thus

$$\operatorname{im} L_{k-1}(M) \oplus \operatorname{im} L_k^*(M) \subseteq (\ker \Delta_k)^\perp = \operatorname{im} \Delta_k.$$

Hence we get that $\operatorname{im} \Delta_k = \operatorname{im} L_{k-1}(M) \oplus \operatorname{im} L_k^*(M)$.

Thus we find that

$$\ker L_k(M) \subseteq (\operatorname{im} L_k^*(M))^\perp = \ker \Delta_k \oplus \operatorname{im} L_{k-1}(M)$$

and reverse containment is clear by Proposition 4.22 and the fact that $L_k \circ L_{k-1} = 0$. Thus we conclude that

$$\ker L_k(M) = \ker \Delta_k \oplus \operatorname{im} L_{k-1}(M)$$

and the theorem follows by definition of $H^k(E_\bullet, L_\bullet)$. □

4.3 Hodge decomposition for flat connections

4.3.1 Elliptic differentials and λ-D-connections

We may now return to the case of interest. Let X be a complex manifold and suppose we have a differential D on the pair $(\mathcal{O}, \mathcal{K}) = (\mathcal{A}^0_{\mathbb{C}}(X), \mathcal{K})$. Note we may view $\mathcal{K}$ as a complex vector bundle K over X having $\mathcal{K}$ as its sheaf of sections. Then, just as in Example 4.8, we have that for $f \in A^0_{\mathbb{C}}(X)$,

$$[D^{(k)}, f]\omega = D^{(k)}(f\omega) - fD^{(k)}(\omega) = D^{(0)} f \wedge \omega$$

so $D^{(k)}$ is a differential operator of order 1. This has symbol

$$\sigma_1(D^{(k)})(x, (\mathrm{d}f)_x)(e) = i(D^{(0)}f)(x) \wedge e$$

and seeing as $D^{(0)}$ gives a $\mathbb{C}$-linear derivation of $\mathcal{A}^0_{\mathbb{C}}(X)_x$ at every $x \in X$, it induces a well-defined map

$$\sigma_D : \mathcal{A}^1_{\mathbb{C}}(X) \longrightarrow \mathcal{K}.$$

making

$$\begin{array}{ccc} \mathcal{A}^0_{\mathbb{C}}(X) & \xrightarrow{D^{(0)}} & \mathcal{K} \\ {\scriptstyle d}\big\downarrow & \nearrow_{\sigma_D} & \\ \mathcal{A}^1_{\mathbb{C}}(X) & & \end{array}$$

commute, for which we can write

$$\sigma_1(D^{(k)})(x, \xi)(e) = i\sigma_{D,x}(\xi) \wedge e.$$

Now, because $D^2 = 0$, we get an associated complex

$$\mathcal{O} \xrightarrow{D} \mathcal{K} \xrightarrow{D} \textstyle\bigwedge^2_{\mathcal{O}} \mathcal{K} \xrightarrow{D} \cdots \xrightarrow{D} \textstyle\bigwedge^n_{\mathcal{O}} \mathcal{K} \tag{4.3.1}$$

where $n = \operatorname{rank} \mathcal{K} = \dim_{\mathbb{C}} K$. We will say that D is *elliptic* if (4.3.1) is an elliptic complex. By the above, we find

Proposition 4.25. *The differential D is elliptic if and only if the composite*

$$(T^*X)_x \hookrightarrow (T^*_{\mathbb{C}}X)_x \xrightarrow{\sigma_{D,x}} K_x$$

is injective for all $x \in X$.

Proof. We have that the symbol complex associated to (4.3.1) at the point (x, ξ) is given by

$$0 \longrightarrow \mathbb{C} \xrightarrow{i\sigma_{D,x}(\xi)\wedge(-)} K_x \xrightarrow{i\sigma_{D,x}(\xi)\wedge(-)} \textstyle\bigwedge^2_{\mathbb{C}} K_x \xrightarrow{i\sigma_{D,x}(\xi)\wedge(-)} \cdots \xrightarrow{i\sigma_{D,x}(\xi)\wedge(-)} \textstyle\bigwedge^n_{\mathbb{C}} K_x \longrightarrow 0$$

which is exact if and only if $\sigma_{D,x}(\xi) \neq 0$. Since this must hold for all $(x, \xi) \in T^*X$ with $\xi \neq 0$, we get the result. □

Remark 4.26. Here we are freely translating between the language of complex vector bundles over X and locally-free $\mathcal{A}^0_{\mathbb{C}}(X)$-modules over X. To convert between these when deciding if D is elliptic one should notice that $\bigwedge^k_{\mathcal{O}} \mathcal{K} = \mathcal{A}^0(\bigwedge^k_{\mathbb{C}} K)$.

Example 4.27. Example 4.19 shows that the exterior derivative d on the pair $(\mathcal{A}^0_{\mathbb{C}}(X), \mathcal{A}^1_{\mathbb{C}}(X))$ is elliptic. Alternatively, we have that $\sigma_d = \mathrm{id}$ so this follows immediately from Proposition 4.25.

Example 4.28 (Dolbeault complex). We have that the differentials ∂ and $\bar{\partial}$ on $(\mathcal{A}^0_{\mathbb{C}}(X), \mathcal{A}^{1,0}(X))$ and $(\mathcal{A}^0_{\mathbb{C}}(X), \mathcal{A}^{0,1}(X))$ respectively are elliptic. Indeed, we have a decomposition

$$\mathcal{A}^1_{\mathbb{C}}(X) = \mathcal{A}^{1,0}(X) \oplus \mathcal{A}^{0,1}(X).$$

If $\pi^{1,0}$ and $\pi^{0,1}$ are the two projections, then we find that $\sigma_\partial = \pi^{1,0}$ and $\sigma_{\bar{\partial}} = \pi^{0,1}$, by definition. Both of these are injective on each fibre of the real cotangent space $T^*X \subseteq T^*_{\mathbb{C}}X$, giving that ∂ and $\bar{\partial}$ are elliptic by Proposition 4.25.

More generally, given a λ-D-connection ∇ on a complex vector bundle E, we have that

$$[\nabla^{(k)}, f](v \otimes \omega) = \lambda v \otimes (Df \wedge \omega)$$

so each $\nabla^{(k)}$ is

- a differential operator of order 0 if $\lambda = 0$
- a differential operator of order 1 if $\lambda \neq 0$.

Additionally, when $\lambda \neq 0$, we can compute the symbols of the $\nabla^{(k)}$ just as in Example 4.12 to find

$$\sigma_1(\nabla^{(k)})(x,\xi)(v \otimes \omega) = \lambda v \otimes \sigma_1(D^{(k)})(x,\xi)(\omega)$$

Now, if ∇ is a flat connection, we get a complex

$$\mathcal{A}^0(E) \xrightarrow{\nabla^{(0)}} \mathcal{A}^0(E) \otimes_{\mathcal{O}} \mathcal{K} \xrightarrow{\nabla^{(1)}} \mathcal{A}^0(E) \otimes_{\mathcal{O}} \bigwedge\nolimits^2_{\mathcal{O}} \mathcal{K} \xrightarrow{\nabla^{(2)}} \cdots \xrightarrow{\nabla^{(n-1)}} \mathcal{A}^0(E) \otimes_{\mathcal{O}} \bigwedge\nolimits^n_{\mathcal{O}} \mathcal{K} \tag{4.3.2}$$

where $n = \operatorname{rank} \mathcal{K}$. This has symbol complex

$$0 \longrightarrow \pi^* E \otimes_{\mathbb{C}} \pi^* \underline{\mathbb{C}} \xrightarrow{\lambda \,\mathrm{id} \otimes \sigma(D^{(0)})} \pi^* E \otimes_{\mathbb{C}} \pi^* K \xrightarrow{\lambda \,\mathrm{id} \otimes \sigma(D^{(1)})} \cdots \longrightarrow \pi^* E \otimes_{\mathbb{C}} \pi^* \bigwedge\nolimits^n_{\mathbb{C}} K \longrightarrow 0.$$

Evaluating fibre-wise, we see that if D is elliptic, then (4.3.2) is elliptic.

Example 4.29. Given a holomorphic vector bundle $(E, \bar{\partial}_E)$ over X, seeing as by Example 4.28 that $\bar{\partial}$ is elliptic, the flat 1-$\bar{\partial}$-connection $\bar{\partial}_E$ gives an elliptic complex.

4.3.2 The holomorphic case

The above section discussed flat λ-D-connections and identified a case in which the Hodge Theorem 4.24 applies when $\lambda \neq 0$. This leaves open the case of $\lambda = 0$. Moreover, the Hodge Theorem 4.24 is smooth in nature, and thus only allowed us to apply it to differentials on $\mathcal{K}$ where $\mathcal{K}$ is a locally free sheaf of $\mathcal{A}^0_{\mathbb{C}}(X)$-modules. However, we would also like to work in the holomorphic category and apply these results for differentials on locally free sheaves of $\mathcal{O}_X$-modules, as well as to associated connections. The goal of this section is to overcome both these limitations at once.

Let E be a complex vector bundle over X and suppose we have two connections on a complex vector bundle E:

- a flat 1-$\bar{\partial}$-connection $\bar{\partial}_E$ on E, turning E into a complex vector bundle
- a flat λ-∂-connection ∇

such that

$$\bar{\partial}_E \nabla + \nabla \bar{\partial}_E = 0.$$

Then we have that $L = \bar{\partial}_E + \nabla$ is not a connection, but it is still a differential operator of order 1 with $L^2 = 0$. Moreover, one computes

$$\sigma_1(L)(x,\xi)(v \otimes w) = iv \otimes (\pi^{0,1}\xi + \lambda \pi^{1,0}\xi) \wedge \omega$$

and thus we have an elliptic complex

$$\mathcal{A}^0(E) \xrightarrow{L} \mathcal{A}^1(E) \xrightarrow{L} \mathcal{A}^2(E) \xrightarrow{L} \cdots \xrightarrow{L} \mathcal{A}^{2n}(E). \tag{4.3.3}$$

where $2n = \dim_{\mathbb{R}} X$. Now, seeing as $\bar{\partial}_E$ and ∇ anti-commute, we get that ∇ descends to a λ-d_h-connection on the holomorphic vector bundle $\mathcal{E} = \ker(\bar{\partial}_E : \mathcal{A}^0(E) \to \mathcal{A}^1(E))$. If Ω^p_X is the sheaf of holomorphic p-forms on X, then we thus we have a complex

$$\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_X \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^n_X. \tag{4.3.4}$$

Proposition 4.30. *The complexes* (4.3.3) *and* (4.3.4) *are quasi-isomorphic.*

Proof. We have a double complex

$$\begin{array}{ccccccccc}
\mathcal{E} & \xrightarrow{\nabla} & \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X & \xrightarrow{\nabla} & \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_X & \xrightarrow{\nabla} & \cdots & \xrightarrow{\nabla} & \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^n_X \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\mathcal{A}^{0,1}(E) & \xrightarrow{\nabla} & \mathcal{A}^{1,1}(E) & \xrightarrow{\nabla} & \mathcal{A}^{2,1}(E) & \xrightarrow{\nabla} & \cdots & \xrightarrow{\nabla} & \mathcal{A}^{n,1}(E) \\
\downarrow \bar{\partial}_E & & \downarrow -\bar{\partial}_E & & \downarrow \bar{\partial}_E & & & & \downarrow (-1)^n\bar{\partial}_E \\
\mathcal{A}^{0,2}(E) & \xrightarrow{\nabla} & \mathcal{A}^{1,2}(E) & \xrightarrow{\nabla} & \mathcal{A}^{2,2}(E) & \xrightarrow{\nabla} & \cdots & \xrightarrow{\nabla} & \mathcal{A}^{n,2}(E) \\
\downarrow \bar{\partial}_E & & \downarrow -\bar{\partial}_E & & \downarrow \bar{\partial}_E & & & & \downarrow (-1)^n\bar{\partial}_E \\
\vdots & & \vdots & & \vdots & & & & \vdots
\end{array}$$

where the columns are resolutions of the $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_X$ by the $\bar{\partial}_E$-Poincaré lemma. Thus we are done by Proposition 2.5 seeing as $(\mathcal{A}^\bullet(E), L)$ is the simple complex associated to the double complex $(\mathcal{A}^{p,q}(E), \nabla, (-1)^p\bar{\partial}_E)$. □

Thus, as (4.3.3) is elliptic, we may use the Hodge Theorem 4.24 to compute the cohomology of (4.3.4). Note also that this process is reversible: Given a holomorphic vector bundle $(\mathcal{E}, \nabla)$ with flat λ-d_h-connection ∇, we can view $\mathcal{E}$ as a complex vector bundle $(E, \bar{\partial}_E)$ with flat 1-$\bar{\partial}$-connection $\bar{\partial}_E$ and lift ∇ to a λ-∂-connection on E which anti-commutes with $\bar{\partial}_E$. Thus we have imported the tools of the Hodge Theorem 4.24 to holomorphic vector bundles $(\mathcal{E}, \nabla)$ with flat λ-d_h-connections for all $\lambda \in \mathbb{C}$.

4.4 The case of Kähler manifolds

We finally return to the case in which (X, ω) is Kähler manifold. Fix a smooth complex vector bundle E over X. The goal here will be to consider certain families $(\mathcal{E}_\lambda, \nabla_\lambda)$ of holomorphic structures $\mathcal{E}_\lambda$ on E with λ-d_h-connections ∇_λ. Provided that the pairs $(\mathcal{E}_\lambda, \nabla_\lambda)$ vary appropriately, we will be able to relate the Laplacians occurring in the associated elliptic complex (4.3.3) discussed in Section 4.3.2 in such a way that, after applying the Hodge Theorem 4.24, gives invariance of $\mathbb{H}^k(X, (\mathcal{E}_\lambda \otimes_{\mathcal{O}_X} \Omega^\bullet_X, \nabla_\lambda))$ as λ varies.

4.4.1 The Chern connection

Suppose we have a flat 1-d-connection ∇ on a complex vector bundle E over X. Then we may decompose $\nabla = \nabla^{1,0} + \nabla^{0,1}$ into its $(1,0)$ and $(0,1)$-components. Then, $\nabla^{1,0}$ is a flat 1-∂-connection on E and $\nabla^{0,1}$ is a flat 1-$\bar{\partial}$-connection on E. Moreover, we have that

$$\nabla^{1,0}\nabla^{0,1} + \nabla^{0,1}\nabla^{1,0} = 0$$

by flatness of ∇. Thus flat 1-d-connections arise as constructions of the form considered in Section 4.3.2. The Chern connection of a holomorphic vector bundle gives a partial converse to this observation.

Proposition 4.31. *Let $(E, \bar{\partial}_E)$ be a holomorphic vector bundle over X with Hermitian metric h. Then there exists a unique 1-d-connection ∇ on E, called the* Chern connection *of $(E, \bar{\partial}_E)$ with respect to h, such that*

(i) given sections σ, τ of E we have $d\langle \sigma, \tau\rangle_h = \langle \nabla\sigma, \tau\rangle_h + \langle \sigma, \nabla\tau\rangle_h$, i.e. ∇ is unitary

(ii) $\nabla^{0,1} = \bar{\partial}_E$

Moreover, $\nabla^{1,0}$ is a flat 1-∂-connection on E.

Remark 4.32. In (i), writing $\langle\ ,\ \rangle_h$ in taken to mean applying $\langle\ ,\ \rangle_h$ on the E-components, i.e. $\langle \omega \otimes e_1, e_2\rangle_h = \langle e_1, e_2\rangle_h \omega$ and $\langle e_1, \omega \otimes e_2\rangle_h = \langle e_1, e_2\rangle_h \bar{\omega}$. Note the conjugation of the 1-form component when it occurs as the right entry due to the sesquilinearity of h.

Proof. For uniqueness, taking the $(1,0)$-part of the metric compatibility constraint (i), we get that

$$\partial\langle \sigma, \tau\rangle_h = \langle \nabla^{1,0}\sigma, \tau\rangle_h + \langle \sigma, \bar{\partial}_E\tau\rangle_h.$$

Thus, taking a local holomorphic frame $\{\sigma_i\}_i$ for E, we get that

$$\langle \nabla^{1,0}\sigma_i, \sigma_j\rangle_h = \partial\langle \sigma_i, \sigma_j\rangle_h.$$

Seeing as h is non-degenerate this uniquely determines $\nabla^{1,0}$. Moreover, one checks such a definition defines a 1-∂-connection $\nabla^{1,0}$ and setting $\nabla = \nabla^{1,0} + \bar{\partial}_E$ gives existence.

As for the flat-ness of $\nabla^{1,0}$, we have that locally

$$\langle (\nabla^{1,0} \circ \nabla^{1,0})\sigma_i, \sigma_j\rangle_h = \partial^2\langle \sigma_i, \sigma_j\rangle_h = 0$$

so $\nabla^{1,0} \circ \nabla^{1,0} = 0$. □

One can also do the same with flat 1-∂-connections on E.

Corollary 4.33. *Let (E, ∂_E) be a complex vector bundle over X with flat 1-∂-connection ∂_E and Hermitian metric h. Then there exists a unique 1-d-connection ∇ on E such that*

(i) ∇ *is unitary*

(ii) $\nabla^{1,0} = \partial_E$.

Moreover, $\nabla^{0,1}$ *is a flat* 1-$\bar{\partial}$*-connection on* E.

Proof. We have that h induces a Hermitian metric on $\overline{E}$ and ∂_E induces a holomorphic structure $\bar{\partial}_E$ on $\overline{E}$. Then apply Proposition 4.31 and conjugate back. □

4.4.2 Harmonic families

Let $(\mathcal{E}_1, \nabla_1)$ be a holomorphic vector bundle with a flat 1-d_h-connection ∇_1, with associated $\mathcal{C}^\infty$-bundle $(E, \bar{\partial}_E)$. Choose a Hermitian metric h on E. By Proposition 4.31, we get a flat 1-∂-connection δ_h such that $\bar{\partial}_E + \delta_h$ is a unitary connection and by Corollary 4.33 we get a flat 1-$\bar{\partial}$-connection δ'_h such that $\nabla_1 + \delta'_h$ is a unitary connection.

Interpolating between these, we get a family

$$\begin{aligned} \bar{\partial}_{E,\lambda} &= \frac{(1+\lambda)\bar{\partial}_E + (1-\lambda)\delta'_h}{2} && (1\text{-}\bar{\partial}\text{-connection}) \\ \nabla_\lambda &= \frac{(1+\lambda)\nabla_1 + (\lambda-1)\delta_h}{2} && (\lambda\text{-}\partial\text{-connection}) \end{aligned} \tag{4.4.1}$$

of connections. A priori, these have no reason to be flat connections, nor anti-commuting. However, when this is the case, we get a family $(\mathcal{E}_\lambda, \nabla_\lambda)$ of holomorphic bundles (with constant underlying complex bundle E) with flat λ-d_h-connections ∇_λ. We give these families a special name.

Definition 4.34. Given a complex vector bundle E over X, a family $(\mathcal{E}_\lambda, \nabla_\lambda)$ is called *harmonic* if there exists a Hermitian connection h on E such that $(\mathcal{E}_\lambda, \nabla_\lambda)$ arises from the above construction applied to $(\mathcal{E}_1, \nabla_1)$. In such a case, we call h a harmonic metric.

4.4.3 Kähler identities and the Abelian Hodge theorem

Our goal is to show that when X is compact Kähler and $(\mathcal{E}_\lambda, \nabla_\lambda)$ is a harmonic family, then $\mathbb{H}^k(X, (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_X, \nabla_\lambda))$ is independent of λ. To do this, given a harmonic family, we may consider the differential operators

$$D_\lambda = \bar{\partial}_{E,\lambda} + \nabla_\lambda.$$

Then, by Proposition 4.30, $(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_X, \nabla_\lambda)$ is quasi-isomorphic to the elliptic complex

$$\mathcal{A}^0(E) \xrightarrow{D_\lambda} \mathcal{A}^1(E) \xrightarrow{D_\lambda} \mathcal{A}^2(E) \xrightarrow{D_\lambda} \cdots \xrightarrow{D_\lambda} \mathcal{A}^{2n}(E). \tag{4.4.2}$$

To apply our Hodge Theorem 4.24 effectively, we will need to get a handle on the D_λ and their adjoints D^*_λ. When considering adjoints in this section, the vector bundles $E \otimes_{\mathbb{C}} \Omega^k_{X,\mathbb{C}}$ will be given the Hermitian metric induced by (c.f. Section 4.2.1) the harmonic metric h on E and the Kähler metric on X.

First, observe that we may write

$$D_\lambda = D_0 + \lambda D'$$

where

$$D' = \frac{\bar{\partial}_E + \nabla_1 + \delta_h - \delta_h'}{2}.$$

A quick exercise shows that $D_0^2 = (D')^2 = 0$ and D' and D_0 are anti-commuting. From this it also follows that the D_λ are pairwise anti-commuting. Next, letting Λ be the adjoint of the Lefschetz operator

$$\begin{aligned} L : \mathcal{A}_{\mathbb{C}}^m(X) &\longrightarrow \mathcal{A}_{\mathbb{C}}^{m+2}(X) \\ \alpha &\longmapsto \alpha \wedge \omega \end{aligned}$$

then, as noted in [12], we have generalizations of the usual Kähler identities for these operators.

Lemma 4.35 (Kähler identities)**.** *We have that* $(D')^* = i[\Lambda, D_0]$ *and* $D_0^* = -i[\Lambda, D']$.

Proof. Follow the approach as in [8] for the traditional case of $D_0 = \bar{\partial}$, $D' = \partial$. See also [11, Lemma 3.1]. □

Corollary 4.36. *Letting* $\Delta_\lambda = D_\lambda D_\lambda^* + D_\lambda^* D_\lambda$ *we have that* $\Delta_\lambda = (1 + |\lambda|^2)\Delta_0$.

Proof. First, observe that

$$(D')^* D_0 = i[\Lambda, D_0]D_0 = -iD_0\Lambda D_0 = -D_0(D')^*$$

by using Lemma 4.35 and the fact that $D_0^2 = 0$. Taking adjoints we get similarly that

$$D_0^* D' + D' D_0^* = 0.$$

Thus we have

$$\begin{aligned} \Delta_\lambda &= (D_0 + \lambda D')(D_0 + \lambda D')^* + (D_0 + \lambda D')^*(D_0 + \lambda D') \\ &= \Delta_0 + \lambda(D'D_0^* + D_0^* D') + \bar{\lambda}(D_0(D')^* + (D')^* D_0) + |\lambda|^2 \Delta_{D'} \\ &= \Delta_0 + |\lambda|^2 \Delta_{D'}. \end{aligned}$$

so we must show $\Delta_{D'} = D'(D')^* + (D')^* D' = \Delta_0$.

For this, we have that

$$\begin{aligned}
i\Delta_0 &= D_0[\Lambda, D'] + [\Lambda, D']D_0 \\
&= D_0\Lambda D' - D_0 D'\Lambda + \Lambda D' D_0 - D'\Lambda D_0 \\
&= D_0\Lambda D' + D' D_0\Lambda - \Lambda D_0 D' - D'\Lambda D_0 \\
&= -D'[\Lambda, D_0] - [\Lambda, D_0]D' \\
&= i\Delta_{D'}
\end{aligned}$$

again using the Kähler identities. □

Corollary 4.37. *Let $\omega \in A^k(E)$ be a form which is both D_λ and D_0-closed and either D_λ or D_0-exact. Then there exists a $\chi \in A^{k-2}(E)$ such that $\omega = D_0 D_\lambda \chi$.*

Proof. We prove the case when ω is D_0-exact. The other case follows similarly. Write $\omega = D_0\alpha$ for some α. Seeing as

$$A^k(E) = \ker \Delta_0 \oplus \operatorname{im} \Delta_0 = \ker \Delta_0 \oplus \operatorname{im} \Delta_\lambda,$$

we may write $\alpha = \gamma + \Delta_\lambda \beta$ with γ Δ_0-harmonic. Then

$$D_0\alpha = D_0\Delta_\lambda\beta = \Delta_\lambda D_0\beta$$

seeing as D_λ and D_λ^* both anti-commute with D_0, so Δ_λ commutes with D_0. Then as $D_\lambda\omega = 0$, we get that

$$D_\lambda\Delta_\lambda D_0\beta = 0 \implies \Delta_\lambda D_0\beta = D_\lambda D_\lambda^* D_0\beta.$$

Thus

$$\omega = D_\lambda D_\lambda^* D_0\beta = D_0 D_\lambda D_\lambda^*\beta$$

so we are done. □

This has a striking consequence.

Theorem 4.38 (Abelian Hodge Theorem)**.** *Let $(\mathcal{E}_\lambda, \nabla_\lambda)$ be a harmonic family over a compact Kähler manifold X. Then the projection*

$$\textstyle\bigcap_{\mu\in\mathbb{C}} \ker D_\mu / \bigcap_{\mu\in\mathbb{C}} \operatorname{im} D_\mu \xrightarrow{\cong} H^k(\mathcal{A}^\bullet(E), D_\lambda) \cong \mathbb{H}^k(X, (\mathcal{E}_\lambda \otimes_{\mathcal{O}_X} \Omega_X^\bullet, \nabla_\lambda)).$$

is an isomorphism for all $\lambda \in \mathbb{C}$.

Proof. Indeed, we have a natural projection

$$\bigcap_{\mu\in\mathbb{C}} \ker D_\mu \longrightarrow H^k(\mathcal{A}^\bullet(E), D_\lambda). \tag{4.4.3}$$

By the Hodge Theorem 4.24, we have that every class in $H^k(\mathcal{A}^\bullet(E), D_\lambda)$ has a Δ_λ-harmonic representative α. But if $\Delta_\lambda\alpha = 0$, then $\Delta_\mu\alpha = 0$ for all μ by Corollary 4.36, and hence α is D_μ-closed for all μ. Hence the map (4.4.3) is surjective.

Now, (4.4.3) has kernel

$$\operatorname{im} D_\lambda \cap \bigcap_{\mu\in\mathbb{C}} \ker D_\mu$$

which by Corollary 4.37 is equal to $\bigcap_{\mu\in\mathbb{C}} \operatorname{im} D_\mu$, completing the result. □

Remark 4.39. Given that we have $D_\lambda = D_0 + \lambda D'$, one checks easily that

$$\bigcap_{\mu\in\mathbb{C}} \ker D_\mu / \bigcap_{\mu\in\mathbb{C}} \operatorname{im} D_\mu = (\ker D_0 \cap \ker D_1)/(\operatorname{im} D_0 \cap \operatorname{im} D_1).$$

Moreover, seeing as $D_\lambda = D_0+\lambda(D_1-D_0)$, the flat bundle $(\mathcal{E}_1, \nabla_1)$ and the Higgs bundle $(\mathcal{E}_0, \nabla_0)$ are enough to recover the whole family $(\mathcal{E}_\lambda, \nabla_\lambda)$. For this reason in literature it is common to consider only the pairs of flat bundles and Higgs bundles related in this matter rather than the interpolating family all together.

An exposition of this result relating the hypercohomologies at $\lambda = 1$ and $\lambda = 0$ (i.e. the hypercohomologies of the flat bundle $(\mathcal{E}_1, \nabla_1)$ and Higgs bundle $(\mathcal{E}_0, \nabla_0)$) from the perspective of non-abelian Hodge theory can be found in [12]. The content of Theorem 4.38 first appeared in [4] where a more algebraic approach can be found.

4.4.4 Consequences

From Theorem 4.38 we may derive many classical abelian Hodge theory results as corollaries. First, as a full circle moment, we may rigorously address the observation of Section 4.1.

Corollary 4.40. *Let X be a compact Kähler manifold. Then there is a decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}$$

where $H^{p,q}$ is the space of classes representable by closed (p, q)-forms. Moreover, $H^{p,q} \cong H^q(X, \Omega^p_X)$. In particular, if $h^{p,q} = \dim_{\mathbb{C}} H^q(X, \Omega^p_X)$ then $h^{p,q} = h^{q,p}$.

Proof. Apply Theorem 4.38 with to the harmonic family $(\mathcal{O}_X, \lambda \cdot \partial)$ which can be seen to be harmonic with respect to the usual Euclidean metric on $\underline{\mathbb{C}}$ which has flat 1-d-connection d.

Note that in this case

$$\bigcap_{\mu\in\mathbb{C}} \ker D_\mu = \ker \partial \cap \ker \bar{\partial}$$

and if $\alpha \in A^k(X)$ is a k-form with pure type decomposition $\alpha = \sum_{p,q} \alpha^{p,q}$, then

$$\begin{aligned} \alpha \in \ker \partial \cap \ker \bar{\partial} &\iff \partial \alpha^{p,q} = \bar{\partial} \alpha^{p,q} = 0 \text{ for all } p, q \\ &\iff d\alpha^{p,q} = 0 \text{ for all } p, q. \end{aligned}$$

Thus

$$\bigcap_{\mu \in \mathbb{C}} \ker D_\mu = \bigoplus_{p+q=k} (\ker d \cap A^{p,q}(X))$$

giving the first part after applying the projection in Theorem 4.38 with $\lambda = 1$. To see that $H^{p,q} \cong H^q(X, \Omega_X^p)$, apply the projection when $\lambda = 0$ and observe that $\ker d \cap A^{p,q}(X)$ gets sent to $H^p(X, \Omega_X^p)$ under the isomorphism

$$H^k(\mathcal{A}^\bullet(X), \bar{\partial}) \cong \mathbb{H}^k(X, (\Omega_X^\bullet, 0)) \cong \bigoplus_{p+q=k} H^q(X, \Omega_X^p). \qquad \square$$

Corollary 4.41. *If X is a compact Kähler manifold, then the odd Betti numbers $\beta_{2k+1}(X)$ of X are even.*

Proof. We have by Corollary 4.40 that

$$\beta_{2k+1}(X) = \sum_{p+q=2k+1} h^{p,q} = 2 \sum_{\substack{p+q=2k+1 \\ p<q}} h^{p,q}. \qquad \square$$

In the case of Corollary 4.40, what made the decomposition so striking is that at some λ we had $\nabla_\lambda = 0$. Analyzing which families this may occur for, we are lead to a slightly more general version of Corollary 4.40 which is also sometimes called the abelian Hodge theorem.

Corollary 4.42. *Let X be a compact Kähler manifold and (E, ∇) a complex vector bundle with flat 1-d-connection ∇. If there exists a Hermitian metric on E such that ∇ is unitary, then we have a canonical decomposition*

$$H^k(X, \mathcal{E}^\nabla) = \bigoplus_{p+q=k} H^{p,q}$$

where $\mathcal{E}^\nabla = \ker(\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E))$ and $H^{p,q}$ is the space of forms representable by ∇-closed (p,q)-forms. Moreover, $H^{p,q} \cong H^q(X, \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^p)$ where $\mathcal{E}$ is the holomorphic vector bundle associated to $(E, \nabla^{0,1})$.

Proof. Suppose that h is Hermitian metric on E such that ∇ is unitary. Since ∇ is a unitary connection, we have that ∇ is the Chern connection with respect to h of the holomorphic vector bundle $(E, \nabla^{0,1})$. Thus $\delta_h = \nabla^{1,0}$ and $\delta_h' = \nabla^{0,1}$ and we have that $(\mathcal{E}, \lambda \cdot \nabla^{1,0})$ is a harmonic family. The rest of the proof is then the same as Corollary 4.40. $\square$

5 Non-Abelian Hodge theory

5.1 Harmonic bundles and overview of strategy

Having seen the abelian Hodge theorem 4.38, classifying harmonic families becomes an immediate interest. By Remark 4.39, rather than considering harmonic families all together, we will switch to focusing on the endpoints $\lambda = 1$ and $\lambda = 0$ which are flat bundles and Higgs bundles respectively and ask when an interpolating family of λ-connections exists.

Let us review the construction of harmonic families from Section 4.4.2. We start with a flat bundle (E, ∇) and break ∇ into its $(1,0)$ and $(0,1)$-components, writing $\nabla = \nabla^{1,0} + \nabla^{0,1}$. Then, for any Hermitian metric h on E, we then may find a flat 1-∂-connection and 1-$\bar{\partial}$-connection δ_h and δ'_h, respectively, such that $\nabla^{0,1} + \delta_h$ and $\nabla^{1,0} + \delta'_h$ are unitary connections with respect to h. We may then define

$$D_0 = \underbrace{\frac{\nabla^{0,1} + \delta'_h}{2}}_{1\text{-}\bar{\partial}\text{-connection}} + \underbrace{\frac{\nabla^{1,0} - \delta_h}{2}}_{0\text{-}\partial\text{-connection}}.$$

This is the data of a Higgs bundle structure on E, provided the 1-$\bar{\partial}$-connection and 0-∂-connection are both flat and anti-commute (c.f. Example 3.9). This is captured by the vanishing of the *pseudo-curvature*[2]

$$\mathbf{G}_h = D_0^2.$$

Conversely, suppose we start with a Higgs bundle $(E, \bar{\partial}_E, \theta)$. Then, for any Hermitian metric h on E, we may find a flat 1-∂-connection δ''_h such that $\bar{\partial}_E + \delta''_h$ is a unitary connection. Moreover, θ has a formal adjoint $\theta^\dagger_h$ (see Appendix A) which is a 0-$\bar{\partial}$-connection satisfying

$$\langle \theta e_1, e_2 \rangle_h = \langle e_1, \theta^\dagger_h e_2 \rangle_h$$

for all sections e_1, e_2 of E. Here $\langle\ ,\ \rangle_h$ denotes applying h to the E-component and leaving the 1-form component unchanged, as in Remark 4.32. Then we may form

$$D_1 = \underbrace{\bar{\partial}_E + \theta^\dagger_h}_{1\text{-}\bar{\partial}\text{-connection}} + \underbrace{\theta + \delta''_h}_{1\text{-}\partial\text{-connection}}$$

which is a 1-d-connection on E. This is a flat connection when the curvature

$$\mathbf{F}_h = D_1^2$$

vanishes.

These constructions are inverse, in a sense. If we have a flat bundle (E, ∇) and Hermitian metric h on E such that the pseudo-curvature $\mathbf{G}_h = 0$, then running the above construction to the resulting Higgs bundle (E, D_0) with the Hermitian metric h recovers (E, ∇). Similarly, if we

[2] The reason for the name pseudo-curvature is that D_0 is not a connection.

start with a Higgs bundle (E, D_0) and Hermitian metric h such that $\mathbf{F}_h = 0$, then applying the inverse construction to the resulting flat bundle (E, D_1) with h recovers (E, D_0).

We show this for the direction starting with a flat bundle. Indeed, in this case,

$$\delta_h'' = \frac{\nabla^{1,0} + \delta_h}{2}$$

since

$$\left(\frac{\nabla^{0,1} + \delta_h'}{2}\right) + \left(\frac{\nabla^{1,0} + \delta_h}{2}\right) = \left(\frac{\nabla^{0,1} + \delta_h}{2}\right) + \left(\frac{\nabla^{1,0} + \delta_h'}{2}\right)$$

is a sum of two unitary connections, hence unitary. Similarly, taking the $(1,0)$ and $(0,1)$-components of the unitary conditions for $\nabla^{0,1} + \delta_h$ and $\nabla^{1,0} + \delta_h'$, we get

$$\begin{aligned}
\bar{\partial}\langle e_1, e_2\rangle_h &= \langle \nabla^{0,1} e_1, e_2\rangle_h + \langle e_1, \delta_h e_2\rangle_h \\
\bar{\partial}\langle e_1, e_2\rangle_h &= \langle \delta_h' e_1, e_2\rangle_h + \langle e_1, \nabla^{1,0} e_2\rangle_h \\
\partial\langle e_1, e_2\rangle_h &= \langle \delta_h e_1, e_2\rangle_h + \langle e_1, \nabla^{0,1} e_2\rangle_h \\
\partial\langle e_1, e_2\rangle_h &= \langle \nabla^{1,0} e_1, e_2\rangle_h + \langle e_1, \delta_h' e_2\rangle_h .
\end{aligned}$$

Combining these, one shows

$$\theta_h^{\dagger} = \frac{\nabla^{0,1} - \delta_h'}{2}.$$

Thus $\nabla^{0,1} = \bar{\partial}_E + \theta_h^{\dagger}$ and $\nabla^{1,0} = \theta + \delta_h''$ and the constructions are inverse in this case. Checking the composite in the other direction is similar.

This motivates the following definition.

Definition 5.1. A *harmonic bundle* is a tuple (E, D_0, D_1), where D_0 is a Higgs bundle structure on E and D_1 is a flat bundle structure on E, such that there exists a Hermitian metric h on E relating D_0 and D_1 by the constructions above.

Remark 5.2. By definition, $(\mathcal{E}_\lambda, \nabla_\lambda)$, where $\mathcal{E}_\lambda$ corresponds to $(E, \bar{\partial}_{E,\lambda})$, is a harmonic family if and only if (E, D_0, D_1) is a harmonic bundle, where $D_\lambda = \bar{\partial}_{E,\lambda} + \nabla_\lambda$. Conversely, (E, D_0, D_1) is a harmonic bundle if and only if $D_\lambda = D_0 + \lambda(D_1 - D_0)$ describes a harmonic family.

We may turn harmonic bundles into a category by asking that morphisms be bundle morphisms intertwining both D_0 and D_1. We call this category $\mathbf{HBun}_X$. Defining the categories $\mathbf{Higgs}_X$ and $\mathbf{FlatBun}_X$ of Higgs bundle and flat bundles over X similarly, we immediately get two forgetful functors

$$\begin{array}{ccc}
 & \mathbf{HBun}_X & \\
 {\scriptstyle (E,D_0,D_1)\mapsto(E,D_1)} \swarrow & & \searrow {\scriptstyle (E,D_0,D_1)\mapsto(E,D_0)} \\
\mathbf{FlatBun}_X & & \mathbf{Higgs}_X .
\end{array}$$

We may also give harmonic bundles a natural notion of cohomology: Seeing as D_0, D_1 anti-commute when (E, D_0, D_1) is a harmonic bundle, D_0 preserves $\ker(D_1 - D_0)$ and we have a complex

$$(\ker(D_1 - D_0 : \mathcal{A}^\bullet(E) \to \mathcal{A}^{\bullet+1}(E)), D_0) = (\ker(D_1 - D_0 : \mathcal{A}^\bullet(E) \to \mathcal{A}^{\bullet+1}(E)), D_1).$$

If one define the cohomology of (E, D_0, D_1) to be the hypercohomology of this complex, then we have

Proposition 5.3. *The forgetful functors* $\mathbf{HBun}_X \to \mathbf{FlatBun}_X$ *and* $\mathbf{HBun}_X \to \mathbf{Higgs}_X$ *are cohomology preserving.*

Proof. This follows immediately from the Abelian Hodge Theorem 4.38 and Remark 4.39. □

The goal of the following sections will be to sketch a proof that these forgetful functors are fully faithful and injective on objects, as well as identify their images on objects. Upon doing this, we obtain a cohomology preserving equivalence between full subcategories of $\mathbf{FlatBun}_X$ and $\mathbf{Higgs}_X$.

5.2 Topological obstructions for harmonic bundles

To identify when we can find a Hermitian metric h on a flat bundle and Higgs bundle such that $\mathbf{G}_h$ and $\mathbf{F}_h$, respectively, vanish we begin by thinking about Chern classes.

Given any harmonic bundle (E, D_0, D_1) we know in particular that E supports a flat connection D_1. By Chern-Weil theory, if E is a smooth vector bundle, then for any 1-d-connection ∇ on E, the rational Chern characters of E may be computed as

$$\mathrm{ch}_k(E) = \frac{1}{k!}\left(\frac{i}{2\pi}\right)^k \mathrm{tr}(\underbrace{\nabla^2 \wedge \cdots \wedge \nabla^2}_{k \text{ times}}) \in H^{2k}_{\mathrm{dR}}(X) \otimes \mathbb{C} = H^{2k}(X, \mathbb{C}).$$

In particular, any E occurring as part of a harmonic bundle should have vanishing Chern classes.

It turns out that considering the components of $\mathrm{ch}_2(E)$ along the Kähler class is a useful thing to do. This is due to the following proposition.

Proposition 5.4. *For* $\Omega = \mathbf{F}_h$ *or* $\mathbf{G}_h$*, we have that*

$$\mathrm{tr}(\Omega \wedge \Omega).[\omega]^{n-2} = (n-2)!\left\{\|\Omega\|^2_{L^2} - \|\Lambda\Omega\|^2_{L^2}\right\}.$$

where $n = \dim_{\mathbb{C}} X$.

Proof. First consider the case $\Omega = \mathbf{F}_h$. We have that $D_1 = \bar{\partial}_E + \delta''_h + \theta + \theta^\dagger$. Writing $\nabla = \bar{\partial}_E + \delta''_h$ which is a unitary connection by construction, we have that

$$\mathbf{F}_h = \nabla^2 + (\theta \wedge \theta^\dagger + \theta^\dagger \wedge \theta) + (\delta''_h\theta + \theta\delta''_h) + (\bar{\partial}_E\theta^\dagger + \theta^\dagger\bar{\partial}_E).$$

Seeing as ∇ is a unitary connection, $(\nabla^2)^\dagger = -\nabla^2$ and using that

$$\partial\langle e_1, e_2\rangle = \langle \delta_h'' e_1, e_2\rangle + \langle e_1, \bar{\partial}_E e_2\rangle,$$

one checks

$$\begin{aligned} (\mathbf{F}_h^{1,1})^\dagger &= -\mathbf{F}_h^{1,1} \\ (\mathbf{F}_h^{2,0})^\dagger &= \mathbf{F}_h^{0,2} \end{aligned} \tag{5.2.1}$$

where $\mathbf{F}_h^{p,q}$ denotes the (p,q)-component of $\mathbf{F}_h$. A similar computation shows that the relations (5.2.1) hold for $\mathbf{G}_h$ as well. Thus we prove the result for an arbitrary $\Omega \in A^2(\operatorname{End} E)$ satisfying the relations in (5.2.1).

For this, given the Hermitian bundle $\operatorname{End} E$, we may define a Hermitian form H on $A^k(\operatorname{End} E)$ for $k \leq n$ by

$$H(\alpha, \beta) = i^k \int_X \operatorname{tr}(\alpha \wedge \beta^\dagger) \wedge \omega^{n-2}.$$

Considering that ω is a real $(1,1)$-form, we find that

$$A^k(\operatorname{End} E) = \bigoplus_{p+q=k} A^{p,q}(\operatorname{End} E)$$

is an orthogonal decomposition for H. Moreover, the Leftschetz decomposition (c.f. [15, Prop. 6.22]) is also an orthogonal decomposition for H. Finally, we have that when β is a primitive (in the sense of the Lefschetz decomposition) form of type (p,q), $k = p+q$, then

$$H(\alpha, \beta) = (-1)^{\frac{k(k+1)}{2}} i^{k+q-p} (n-k)! \langle \alpha, \beta\rangle_{L^2} \tag{5.2.2}$$

where the L^2-inner product is defined in terms of the induced Hermitian metric on $\operatorname{End} E$ (see Appendix A and [15, Prop. 6.29]). These are known as the Hodge-Riemann bilinear relations.

Given the relations (5.2.1) hold for Ω, we have that

$$\int_X \operatorname{tr}(\Omega \wedge \Omega) \wedge \omega^{n-2} = H(\Omega, \Omega^{2,0} + \Omega^{0,2} - \Omega^{1,1}).$$

Now, $\Omega^{2,0}$ and $\Omega^{0,2}$ are primitive by degree reasons, and $\Omega^{1,1}$ has Lefschetz decomposition

$$\Omega^{1,1} = (\Omega^{1,1} - \frac{1}{n} L\Lambda\Omega^{1,1}) + \frac{1}{n} L\Lambda\Omega^{1,1}$$

by the Kähler identities. Thus, writing $H(\alpha)$ for $H(\alpha,\alpha)$, we have that

$$\begin{aligned}H(\Omega,\Omega^{2,0}+\Omega^{0,2}-\Omega^{1,1}) &= H(\Omega^{2,0})+H(\Omega^{0,2})-H(\Omega^{1,1}-\frac{1}{n}L\Lambda\Omega^{1,1})-\frac{1}{n^2}H(L\Lambda\Omega^{1,1})\\ &=(n-2)!\left\{\|\Omega^{0,2}+\Omega^{2,0}\|_{L^2}+\|\Omega^{1,1}-\frac{1}{n}L\Lambda\Omega^{1,1}\|_{L^2}-\frac{n-1}{n}\|\Lambda\Omega^{1,1}\|_{L^2}\right\}\\ &=(n-2)!\left\{\|\Omega-\frac{1}{n}L\Lambda\Omega\|_{L^2}-\frac{n-1}{n}\|\Lambda\Omega\|_{L^2}\right\}\end{aligned}$$

where we have used orthogonality, (5.2.2) and the fact that $\Lambda\Omega=\Lambda\Omega^{1,1}$ for degree reasons. Finally, expanding out $\|\Omega-\frac{1}{n}L\Lambda\Omega\|_{L^2}$ using that Λ is the adjoint of L with respect to the L^2-inner product, one completes the proof. □

While the quantity $\mathrm{tr}(\mathbf{F}_h\wedge\mathbf{F}_h)$ has immediate meaning by Chern-Weil theory since $\mathbf{F}_h$ is a curvature form, such an interpretation is less clear for the pseudo-curvature $\mathbf{G}_h$. However, the next proposition shows that $\mathrm{tr}(\mathbf{G}_h\wedge\mathbf{G}_h)$ still has meaning.

Proposition 5.5. *We have that* $\mathrm{tr}(\mathbf{G}_h\wedge\mathbf{G}_h).[\omega]^{n-2}=0$.

Proof. We begin with a flat bundle (E,D_1) and produce D_0 with $\mathbf{G}_h=D_0^2$. This gives an interpolating family $D_\lambda=D_0+\lambda(D_1-D_0)$ of differential operators which are a sum of a 1-$\bar{\partial}$-connection and a λ-∂-connection. Thus rescaling the $(1,0)$-component and setting

$$\nabla_\lambda=D_\lambda^{0,1}+\lambda^{-1}D_\lambda^{1,0}$$

for $\lambda\neq 0$, we see that each ∇_λ is a 1-d-connection. Since E supports the flat connection D_1, its positive degree rational Chern classes are zero. Thus by Chern-Weil theory, we get

$$\begin{aligned}0=\mathrm{ch}_2(E).[\omega]^{n-2} &= \int_X \mathrm{tr}(\nabla_\lambda^2\wedge\nabla_\lambda^2)\wedge\omega^{n-2}\\ &=\lambda^{-2}\int_X\mathrm{tr}(D_\lambda^2\wedge D_\lambda^2)\wedge\omega^{n-2}\end{aligned}$$

where the last line comes from the fact that ω is of type $(1,1)$, allowing us to drop terms for degree reasons. Hence for all $\lambda\neq 0$ we have

$$\int_X\mathrm{tr}(D_\lambda^2\wedge D_\lambda^2)\wedge\omega^{n-2}=0,$$

so letting $\lambda\to 0$ we are done. □

Combining Propositions 5.4 and 5.5 we conclude the following.

(i) Going from flat bundles to Higgs bundles, we have that $\mathbf{G}_h=0$ if and only if $\Lambda\mathbf{G}_h=0$

(ii) Going from Higgs bundles to flat bundles, we have that $\mathbf{F}_h=0$ if and only if $\Lambda\mathbf{F}_h=0$ and $\mathrm{ch}_2(E).[\omega]^{n-2}=0$.

Seeing as having vanishing positive degree rational Chern classes is a topological obstruction to solving $\mathbf{F}_h = 0$, independent of h, we adopt $\mathrm{ch}_2(E).[\omega]^{n-2} = 0$ as an assumption on our Higgs bundles. From this, we then reduce solving $\mathbf{F}_h = 0$ and $\mathbf{G}_h = 0$ to the *a priori* weaker constraints $\Lambda\mathbf{F}_h = 0$ and $\Lambda\mathbf{G}_h = 0$.

5.3 Existence criteria and the non-abelian Hodge theorem

5.3.1 Flat bundles to harmonic bundles

Let (E, D_1) be a flat bundle over X. Having reduced to solving $\Lambda\mathbf{G}_h = 0$, we give metrics h achieving this a name.

Definition 5.6. A *harmonic* metric on a flat bundle (E, D_1) is a Hermitian metric h such that $\Lambda\mathbf{G}_h = 0$.

Classifying when a Higgs bundle may be equipped with a harmonic metric h would then tell us which Higgs bundle (E, D_0) may be extended to a harmonic bundle (E, D_0, D_1). Thankfully, there is a concise answer to this existence problem due to Corlette. The following result of non-linear analysis comprises one half of the non-abelian Hodge correspondence.

Theorem 5.7 (Corlette [3])**.** *A flat bundle admits a harmonic metric if and only if it is semisimple. Moreover, this harmonic metric is unique up to scaling.*

Here, a flat bundle (E, ∇) is semisimple if every subbundle fixed by ∇ has a complementary subbundle fixed by ∇.

Remark 5.8. The statement of Theorem 5.7 as it appears in [3] is stated in terms of the harmonicity of the classifying map

$$\Phi_h : \tilde{X} \to \mathrm{GL}_n(\mathbb{C})/\mathrm{U}(n)$$

associated to h, where $\tilde{X}$ is the universal cover of X. This ultimately is the reason for the adjective "harmonic" appearing in harmonic metric and harmonic bundle. A discussion on the equivalence between these two notions of harmonic metric can be found in the discussion proceeding [12, Lemma 1.1].

5.3.2 Higgs bundles to harmonic bundles

Let (E, D_0) be a Higgs bundle over X. Going from Higgs bundle to harmonic bundles is more delicate. After imposing the topological constraint $\mathrm{ch}_2(E).[\omega]^{n-2} = 0$, we are interested in solving $\Lambda\mathbf{F}_h = 0$. As it turns out, it is better to study the more general equation $\Lambda\mathbf{F}_h = \gamma\,\mathrm{id}_E$.

Definition 5.9. A Hermitian metric h on a Higgs bundle (E, D_0) is called *Hermitian-Yang-Mills* if $\Lambda\mathbf{F}_h = \gamma\,\mathrm{id}_E$ for some constant $\gamma \in \mathbb{C}$.

Luckily, the constant γ is topological in nature as the next proposition shows.

Proposition 5.10. *For a Higgs bundle* (E, D_0) *with Hermitian-Yang-Mills metric* h*, we have*

$$\gamma = -\frac{2\pi i}{(n-1)!\mathrm{vol}(X)} \cdot \frac{\mathrm{ch}_1(E).[\omega]^{n-1}}{\mathrm{rk}(E)}.$$

Proof. On one hand, we have

$$\int_X \mathrm{tr}\,\Lambda\mathbf{F}_h \cdot \frac{\omega^n}{n!} = \gamma\,\mathrm{rk}(E)\mathrm{vol}(X) \tag{5.3.1}$$

since $\Lambda\mathbf{F}_h = \gamma\,\mathrm{id}_E$. On the other hand, in the notation of the proof of Proposition 5.4,

$$\begin{aligned}\frac{1}{n}\int_X \mathrm{tr}\,\Lambda\mathbf{F}_h \cdot \omega^n &= H(-\frac{1}{n}L\Lambda\mathbf{F}_h, L\,\mathrm{id}_E)\\ &= -H(\mathbf{F}_h^{1,1}, L\,\mathrm{id}_E) + H(\mathbf{F}_h^{1,1} - \frac{1}{n}L\Lambda\mathbf{F}_h, L\,\mathrm{id}_E).\end{aligned}$$

But $\mathbf{F}_h^{1,1} - \frac{1}{n}L\Lambda\mathbf{F}_h$ is a primitive $(1,1)$-form, so

$$\begin{aligned}H(\mathbf{F}_h^{1,1} - \frac{1}{n}L\Lambda\mathbf{F}_h, L\,\mathrm{id}_E) &= C\langle\mathbf{F}_h^{1,1} - \frac{1}{n}L\Lambda\mathbf{F}_h, L\,\mathrm{id}_E\rangle_{L^2}\\ &= C\langle\Lambda\mathbf{F}_h, \mathrm{id}_E\rangle_{L^2} - C\langle\frac{1}{n}\Lambda L\Lambda\mathbf{F}_h, \mathrm{id}_E\rangle_{L^2}\\ &= 0\end{aligned}$$

where C is some constant coming from the Hodge-Riemann bilinear relations. Thus

$$\begin{aligned}\int_X \mathrm{tr}\,\Lambda\mathbf{F}_h \cdot \frac{\omega^n}{n!} &= -\frac{1}{(n-1)!}H(\mathbf{F}_h^{1,1}, L\,\mathrm{id}_E)\\ &= \frac{1}{(n-1)!}\int_X \mathrm{tr}\,\mathbf{F_h} \wedge \omega^{n-2}\\ &= -\frac{2\pi i}{(n-1)!}\mathrm{ch}_2(E).[\omega]^{n-1}.\end{aligned} \tag{5.3.2}$$

Combining (5.3.1) and (5.3.2) gives the result. □

The quantities in Proposition 5.10 are important so we give them a name.

Definition 5.11. The *degree* of a vector bundle E over the compact Kähler manifold (X, ω) is

$$\deg_\omega(E) = \mathrm{ch}_1(E).[\omega]^{n-1}.$$

The *slope* of E is defined as

$$\mu(E) = \frac{\deg_\omega(E)}{\mathrm{rk}(E)}.$$

Thus, as every Higgs bundle occurring in a harmonic bundle must have vanishing Chern characters, we see that the condition $\mathbf{F}_h = 0$ is equivalent to the existence of a Hermitian-Yang-

Mills metric and

$$\mathrm{ch}_2(E).[\omega]^{n-2} = \mathrm{ch}_1(E).[\omega]^{n-1} = 0.$$

To state our existence theorem for Hermitian-Yang-Mills metrics, we need some stability conditions on our Higgs bundles.

Definition 5.12. Let $(\mathcal{E}, \phi)$ be a Higgs bundle, where $\mathcal{E}$ is a holomorphic bundle and $\phi : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ is the Higgs field. We say that $\mathcal{E}$ is *stable* if for every holomorphic subbundle $\mathcal{F} \subseteq \mathcal{E}$ preserved[3] by ϕ we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$.

We say that $(\mathcal{E}, \phi)$ is *polystable* if it is the direct sum of of stable Higgs bundles of the same slope.

The second half of the non-abelian Hodge theorem is then given to us by the following theorem, which tells us exactly when Hermitian-Yang-Mills metrics exist.

Theorem 5.13 (Simpson [11])**.** *A Higgs bundle has a Hermitian-Yang-Mills metric if and only if it is polystable. Moreover, such a metric is unique up to scalars.*

Combining Theorems 5.7 and 5.13 we arrive at the statement of the non-abelian Hodge theorem, as promised in the introduction.

Corollary 5.14 (Non-Abelian Hodge Correspondence)**.** *Let (X, ω) be a compact Kähler manifold. There is a cohomology preserving equivalence between*

(i) *semisimple flat bundles (E, ∇) over X*

(ii) *polystable Higgs bundles $(\mathcal{E}, \phi)$ over X with $\mathrm{ch}_1(\mathcal{E}).[\omega]^{n-1} = \mathrm{ch}_2(\mathcal{E}).[\omega]^{n-2} = 0$.*

Remark 5.15. Here, and elsewhere in this essay, we have been ambiguous as to how one should interpret the meaning of "cohomology preserving." While the appropriate interpretation should be clear from the results stated and proven, one may formalize this by viewing the categories $\mathbf{HBun}_X$, $\mathbf{FlatBun}_X$ and $\mathbf{Higgs}_X$ as *differential graded categories*. The equivalences above then becomes equivalences of differential graded categories, from which preservation of cohomology may be interpreted as taking the associated k-th cohomology categories of the two equivalent dg-categories. More on this is contained in [7, 12].

[3] We say ϕ preserves $\mathcal{F}$ if $\phi(\mathcal{F}) \subseteq \mathcal{F} \otimes \Omega^1_X$.

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A Constructions on Hermitian bundles

In this section we review some basic constructions on Hermitian vector bundles over a manifold X. Throughout, $E \to X$ will denote a smooth, complex vector bundle over X, and h will be a Hermitian metric on E.

Dual and tensorial extension

From the Hermitian metric h on E, we may give $E^\vee$ a Hermitian metric $h^\vee$. This is done by asserting that in each fibre E_x of E, if $e_1, \ldots, e_n$ is an orthonormal basis with respect to h_x, then the dual basis $e_1^*, \ldots, e_n^*$ is declared to be an orthonomal basis for $(E^\vee)_x = (E_x)^\vee$. Doing this defines for each $x \in X$ a Hermitian metric $h_x^\vee$ on $(E^\vee)_x$. These metrics vary smoothly in x and define the *induced* metric on $E^\vee$.

Now suppose that we have another Hermitian bundle $F \to X$ with metric k. Then $E \otimes_{\mathbb{C}} F$ comes with an induced Hermitian metric $h \otimes k$, given in each fibre by

$$\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle_{h \otimes k} = \langle e_1, e_2 \rangle_h \cdot \langle f_1, f_2 \rangle_k.$$

This is referred to as *tensorial extension*. One may refer to Section 4.2.1 for a discussion of both these induced metrics in the special case of interest to the broader essay.

Adjoints and extensions to $\operatorname{End} E$

By the above, a Hermitian metric h on E induces metrics on $E^\vee$ and hence on $\operatorname{End} E = E \otimes E^\vee$. Moreover, the metric on E induces an involution

$$(-)^\dagger : \operatorname{End} E \to \operatorname{End} E.$$

Fibre-wise, given $A \in \operatorname{End} E_x$, $A^\dagger$ is defined by insisting that

$$\begin{array}{ccc} \overline{E}_x^\vee & \xrightarrow{\overline{A}^\vee} & \overline{E}_x^\vee \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ E_x & \xrightarrow{A^\dagger} & E_x \end{array}$$

commutes, where the isomorphisms $\overline{E}^\vee \to E$ are induced by the metric h on E. These fibre-wise maps vary smoothly, and define the operator $(-)^\dagger : \operatorname{End} E \to \operatorname{End} E$ which is a bundle morphism of the underlying real vector bundles, and anti-linear on fibres.

The operator $(-)^\dagger$ has two useful properties which may be checked locally in a trivializing neighborhood. Firstly, for two global sections $A, B \in \Gamma(X, \operatorname{End} E)$ we have that

$$\langle A, B \rangle_{\operatorname{End} E} = \operatorname{tr}(AB^\dagger)$$

where the inner product $\langle\ ,\ \rangle_{\operatorname{End} E}$ refers to that given by the induced metric on $\operatorname{End} E$. Secondly, given sections $e_1, e_2 \in \Gamma(X, E)$ and $A \in \Gamma(X, \operatorname{End} E)$ we have that

$$\langle Ae_1, e_2\rangle_h = \langle e_1, A^\dagger e_2\rangle_h.$$

Thus $A^\dagger$ is the *adjoint* of A with respect to the Hermitian metric h. Moreover, given another Hermitian bundle (F, k), we may define

$$(-)^\dagger : \operatorname{End} E \otimes_{\mathbb{C}} F \to \operatorname{End} E \otimes_{\mathbb{C}} F$$

by $(-)^\dagger \otimes \mathrm{id}_F$. In this case, we still have that for $A \in \Gamma(X, \operatorname{End} E \otimes_{\mathbb{C}} F)$ and $e_1, e_2 \in \Gamma(X, E)$ that

$$\langle Ae_1, e_2\rangle_h = \langle e_1, A^\dagger e_2\rangle_h$$

where now $\langle\ ,\ \rangle_h$ is meant to be interpreted as applying $\langle\ ,\ \rangle_h$ to the E-components and leaving the F-components unchanged. This is used in Section 5.1 when taking the adjoints of 0-∂-connections.

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