Introduction to Hochschild Homology

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1 Introduction and motivation

Let A be a (not necessarily commutative) associative k-algebra (where k is a unital commutative ring). A natural question is the following: What is a "universal" commutative algebra that we may associate to A? Categorically, one may formalize this question as asking for adjoints to the inclusion

$$\mathrm{Alg}_k \longrightarrow \mathrm{CAlg}_k.$$

Hochschild (co)homology arises from studying this question in an appropriate setting. More generally, given an A-bimodule M, we may ask for a universal A-bimodule arising from M in which the left and right actions agree. Since we now have M, A and k to keep track of, we make explicit what we mean by an A-bimodule in this context.

Definition 1.1. An A/k-bimodule is a k-module M which also has the structure of a left and right A-module such that $a_1(ma_2) = (a_1m)a_2$.

Given an A/k-bimodule M, let us explore how to equalize the left and right actions. From experience, we can expect to construct this in two manners—either we could take a maximal sub-object on which the actions agree, i.e.

$$Z(M) \coloneqq \{m \in M : \forall a \in A,\ am = ma\},$$

or we could take the smallest quotient forcing agreement, i.e.

$$M/[A, M] \coloneqq M/\langle am - ma : a \in A,\ m \in M\rangle.$$

We can rephrase this categorically as follows. Notice that giving an A/k-bimodule in which the two actions agree is equivalent to giving a (left) $A/[A, A]$-module. Indeed, if M is an A/k-bimodule in which the two actions agree, we have that

$$\begin{aligned}(ab)\cdot m &= m\cdot(ab)\\ &= (m\cdot a)\cdot b\\ &= b\cdot(a\cdot m)\\ &= (ba)\cdot m\end{aligned}$$

so that $[A,A]$ annihilates M, and the left and right actions descend to an $A/[A,A]$-action. Put differently, we have a fully faithful inclusion

$$\iota : A/[A,A]\text{−Mod} \longrightarrow A/k\text{−BiMod}$$

whose essential image is A/k-bimodules whose left and right actions agree. One then verifies the following.

Exercise 1.1. The functors

$$\begin{aligned} A/k\text{−BiMod} &\longrightarrow A/[A,A]\text{−Mod}\\ M &\longmapsto M/[A,M]\end{aligned}$$

and

$$\begin{aligned} A/k\text{−BiMod} &\longrightarrow A/[A,A]\text{−Mod}\\ M &\longmapsto Z(M)\end{aligned}$$

are left and right adjoints to ι, respectively.

Definition 1.2. Let $\mathrm{HH}(A/k,-)$ be the derived functor of

$$\begin{aligned} A/k\text{−BiMod} &\longrightarrow A/[A,A]\text{−Mod}.\\ M &\longmapsto M/[A,M]\end{aligned}$$

The *Hochschild homology* of M is $\mathrm{HH}(A/k,M)$.

Dually, one could consider the derived functors of $M\mapsto Z(M)$ which would produce *Hochschild cohomology*. In this next section, we will address how to compute $\mathrm{HH}(A/k,M)$. We will focus on the case in which A is in fact commutative, so that we may use homological techniques. More generally, to take the appropriate notion of "derived functor" when A is non-commutative, one must use simplicial techniques.

From a purely algebraic standpoint, one may be interested in doing this to get a continuation of the exact sequence

$$M_1/[A, M_1] \longrightarrow M_2/[A, M_2] \longrightarrow M_3/[A, M_3] \longrightarrow 0$$

into a LES given an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of A/k-bimodules.

If we restrict to the case where A is commutative, then this endeavor can also be motivated geometrically. For this, notice that providing an A/k-bimodule structure on M is equivalent to providing a left (or right) $A \otimes_k A$-module structure on M. Indeed, given an A/k-bimodule M we may equip M with a left $A \otimes_k A$-module structure via

$$(a \otimes b) \cdot m = (a \cdot m) \cdot b$$

and a right $A \otimes_k A$-module structure via

$$m \cdot (a \otimes b) = (b \cdot m) \cdot a.$$

If we let $A^e = A \otimes_k A$ (often called the *enveloping algebra*), then under this correspondence we realize that the inclusion

$$A\text{−Mod} \longrightarrow A/k\text{−BiMod} \simeq A \otimes_k A\text{−Mod}$$

is simply given by viewing an A-module M as an A^e-module via the multiplication map $m : A^e \to A$. But this, however, is exactly the pushforward

$$\Delta_* : \mathrm{QC}(\mathrm{Spec}\, A) \longrightarrow \mathrm{QC}(\mathrm{Spec}\, A \times_k \mathrm{Spec}\, A)$$

along the diagonal map Δ. Thus our left adjoint of interest is simply Δ^* and we have the following corollary.

Proposition 1.3. *Given an A/k-bimodule M, $M/[A, M] \cong M \otimes_{A^e} A$ where A is viewed as an A^e-module via the multiplication map.*

Deriving this functor is naturally motivated by perspective that rather than looking at $\mathrm{QC}(X)$ one should look at the derived category $\mathcal{D}_{\mathrm{qc}}(X)$ of quasi-coherent sheaves on X. In this case, Hochschild homology may simply be viewed as the pullback along the diagonal between derived categories of quasi-coherent sheaves.

2 The bar complex and explicit models of HH

From here on out, we assume that A is commutative so that we may utilize the geometric perspective espoused above. The goal of this section is to relate the definition of Hochschild homology given in the above section to the classical definition of Hochschild homology and discuss their relationship.

We saw from above discussion that Hochschild homology is in fact the derived functor of

$$\begin{aligned} \Delta^* : \mathrm{QC}(\operatorname{Spec} A \times_k \operatorname{Spec} A) &\longrightarrow \mathrm{QC}(\operatorname{Spec} A) \\ M &\longmapsto M \otimes_{A^e} A, \end{aligned}$$

so we get the following corollary.

Corollary 2.1. *For an A/k-bimodule, $\mathrm{HH}(A/k, M) \simeq M \otimes^{\mathbf{L}}_{A^e} A$. In particular, we have that $\mathrm{HH}_*(A/k, M) \cong \mathrm{Tor}^{A^e}_*(M, A)$.*

By symmetry of the Tor functor (or really of the derived tensor product), this may be computed with either a projective resolution of M or a projective resolution of A. We discuss how to do the latter when A is a free k-module, e.g. when k is a field.

Definition 2.2. The *standard complex* or *bar complex* of A is the complex of A^e-modules

$$\cdots \xrightarrow{d_2} A \otimes_k A \otimes_k A \xrightarrow{d_1} A \otimes_k A \xrightarrow{m} A \longrightarrow 0$$

where

$$\begin{aligned} d_n(a_0 \otimes a_2 \otimes \cdots \otimes a_{n+1}) = &\sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \\ &+ (-1)^{n+1} a_{n+1} a_0 \otimes a_1 \otimes \cdots \otimes a_n. \end{aligned}$$

Here, $A^{\otimes(n+2)}$ is viewed as an A^e-module via $(\alpha \otimes \beta) \cdot (a_0 \otimes \cdots \otimes a_{n+1}) = \alpha a_1 \otimes \cdots \otimes a_{n+1}\beta$.

When A is a free k-module, this gives a resolution of A by free A^e-modules as the next proposition shows.

Proposition 2.3. *(i) The bar complex is exact.*

(ii) As left A^e-modules, $A^{\otimes(n+2)} \cong A^e \otimes_k A^{\otimes n}$ via

$$\begin{aligned} A^{\otimes(n+2)} &\longrightarrow A^e \otimes_k A^{\otimes n}. \\ a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} &\longmapsto (a_0 \otimes a_{n+1}) \otimes (a_1 \otimes \cdots \otimes a_n) \end{aligned}$$

In particular, if A is free as a k-module, then $A^{\otimes(n+2)}$ is free as an A^e-module.

Proof. For (i), see Weibel. For (ii), recall that $A^{\otimes(n+2)}$ is viewed as an A^e-module via multiplication on the two outermost factors. Bringing them to the front, we get the desired isomorphism. Finally, if $A \cong k^{\oplus I}$ is free as a k-module, then $A^e \otimes_k A^{\otimes n} \cong (A^e)^{\oplus I^n}$ is free as an A^e-module. □

Theorem 2.4. *If A is a free as a k-module, then Hochschild homology of M,* $\mathrm{HH}(A/k, M)$*, is equivalent to*

$$\cdots \xrightarrow{d_3} M \otimes_k A \otimes_k A \xrightarrow{d_2} M \otimes_k A \xrightarrow{d_1} M \longrightarrow 0$$

where

$$d_i(m \otimes a_1 \otimes \cdots \otimes a_n) = ma_1 \otimes a_2 \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$
$$+ (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

Proof. In this setting, we may use the bar complex to give a free resolution of A as an A^e-module. The terms in this resolution are given by

$$\begin{aligned} M \otimes_{A^e} A^{\otimes(n+2)} &\cong M \otimes_{A^e} (A^e \otimes_k A^{\otimes n}) \\ &\cong M \otimes_k A^{\otimes_k}. \end{aligned}$$

Chasing the differentials through these isomorphisms, we get the claimed complex. □

Remark 2.5. In the literature, especially historical literature, one may see Hochschild homology *defined* as in Corollary 2.4. Here we subscribe to the idea that this should really be viewed as an attempt to compute $\mathrm{HH}(A/k, M) \simeq M \otimes^{\mathbf{L}}_{A^e} A$ and that this is the "correct" version. When A is a free k-module these two definitions agree.

3 Computational techniques and HKR

We introduce some notation.

- When $M = A$, we will write $\mathrm{HH}(A/k)$ for $\mathrm{HH}(A/k, M)$.
- We will write $\mathrm{HH}_*(A/k) = H_*(\mathrm{HH}(A/k))$ for the homology groups of $\mathrm{HH}(A/k)$.

The most basic fact is the following, which follows from general results on homological algebra when taking derived functors.

Theorem 3.1. *Let*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

be a SES of A^e-modules. Then we have an associated LES

$$\cdots \longrightarrow \mathrm{HH}_1(A/k, M_2) \longrightarrow \mathrm{HH}_1(A/k, M_3) \longrightarrow \mathrm{HH}_0(A/k, M_1) \longrightarrow \mathrm{HH}_0(A/k, M_2) \longrightarrow \mathrm{HH}_0(A/k, M_3) \longrightarrow 0.$$

Moreover, if M is flat as an A^e-module, then $\mathrm{HH}_i(A/k, M) = 0$ for all $i > 0$.

Example 3.2. Being the derived functor of $M \mapsto M/[A, M]$, we know that $\mathrm{HH}_0(A/k, M) = M/[A, M]$. In particular, if A is a commutative algebra, then

$$\mathrm{HH}_0(A/k) = A/[A, A] = A.$$

When Corollary 2.4 applies, this can also be seen explicitly as $d_1(m\otimes a) = ma - am$ so $\operatorname{im} d_1 = [A, M]$.

Example 3.3. Let us compute $\mathrm{HH}_*(k[x]/k, M)$ for any module M. To do this, we use the fact that $k[x]$ has a shorter free resolution as an $k[x]^e$-module than the standard complex. Indeed, one may check that

$$0 \longrightarrow k[x] \otimes_k k[x] \xrightarrow{(x\otimes 1 - 1\otimes x)\cdot} k[x] \otimes_k k[x] \xrightarrow{m} k[x] \longrightarrow 0$$

is a free resolution of $k[x]$ as an $k[x]^e$-module. Tensoring with M over $k[x]^e$ we get that $\mathrm{HH}(k[x]/k, M)$ is given by the complex

$$\begin{aligned} 0 \longrightarrow M &\longrightarrow M \longrightarrow 0. \\ m &\longmapsto xm - mx \end{aligned}$$

In particular, if $M = k[x]$, then this middle map is zero, so

$$\mathrm{HH}_*(k[x]/k) = \begin{cases} k[x] & * = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

More generally, $k[x_1, \ldots, x_n]$ has a short free resolution as an $k[x_1, \ldots, x_n]^e$-module given by the *Koszul complex*, which allows one to compute $\mathrm{HH}_*(k[x_1, \ldots, x_n]/k)$.

Example 3.4. Consider the SES

$$0 \longrightarrow I \longrightarrow A^e \xrightarrow{m} A \longrightarrow 0 \tag{†}$$

where $m : A^e \to A$ is multiplication and $I = \ker m$. Now, since A^e is a flat A^e-module we have that

$\mathrm{HH}_i(A/k, A^e) = 0$ for all $i > 0$. Thus the LES associated to (†) gives

$$0 \longrightarrow \mathrm{HH}_1(A/k) \longrightarrow I \otimes_{A^e} A \xrightarrow{\varphi} A$$

where

$$\varphi((a \otimes b) \otimes \alpha) = a\alpha b.$$

But I is the kernel of multiplication and A is commutative, so $\varphi = 0$. Thus $\mathrm{HH}_1(A/k) \cong I \otimes_{A^e} A$. But $A \cong A^e/I$ as an A^e-module via the multiplication map, so

$$\mathrm{HH}_1(A/k) \cong I \otimes_{A^e} A^e/I \cong I/I^2.$$

Finally,

$$\begin{aligned} \Omega^1_{A/k} &\xrightarrow{\cong} I/I^2 \\ \mathrm{d}a &\longmapsto a \otimes 1 - 1 \otimes a \end{aligned}$$

and so in particular $\mathrm{HH}_1(A/k) \cong \Omega^1_{A/k}$, the module of Kähler differentials of A/k.

This is in fact the shadow of a more general phenomenon. It turns out that $\mathrm{HH}_*(A/k)$ has the structure of a graded A-algebra. When Theorem 2.4 applies, one may construct this product explicitly using shuffle products. However, this algebra structure can be constructed much more naturally. Let $X = \operatorname{Spec} A$. Then by earlier discussions we have that $\mathrm{HH}(A/k) = \Delta^*\Delta_*\mathcal{O}_X$ (interpreted in a sufficiently derived context). Now Δ^* is monoidal, so its right adjoint Δ_* is lax monoidal, which is to say there exists a natural map

$$\Delta_*\mathcal{F} \otimes \Delta_*\mathcal{G} \to \Delta_*(\mathcal{F} \otimes \mathcal{G}).$$

The multiplication on $\mathrm{HH}(A/k)$ is then given by

$$\Delta^*\Delta_*\mathcal{O}_X \otimes \Delta^*\Delta_*\mathcal{O}_X \cong \Delta^*(\Delta_*\mathcal{O}_X \otimes \Delta_*\mathcal{O}_X) \longrightarrow \Delta^*\Delta_*(\mathcal{O}_X \otimes \mathcal{O}_X) \xrightarrow{\Delta^*\Delta_* m} \Delta^*\Delta_*\mathcal{O}_X.$$

However, for us it is fine to blackbox this.

Theorem 3.5 (Hochschild-Kostant-Rosenberg)**.** *The identification* $\mathrm{HH}_1(A/k) \cong \Omega^1_{A/k}$ *induces a map*

$$\textstyle\bigwedge^\bullet \Omega^1_{A/k} \cong \bigwedge^\bullet \mathrm{HH}_1(A/k) \longrightarrow \mathrm{HH}_\bullet(A/k)$$

via multiplication. When k is a field and A is a smooth k-algebra, this map is an isomorphism.

This is in fact the start of a long story. Thus shows that $\mathrm{HH}_\bullet(A/k)$ is isomorphic in broad settings to the de Rham complex with zero differential. However, it turns out that $\mathrm{HH}_\bullet(A/k)$ has a degree raising differential

$$b : \mathrm{HH}_\bullet(A/k) \to \mathrm{HH}_{\bullet+1}(A/k)$$

which under the HKR isomorphism corresponds to the de Rham differential. Even further, this differential arising from a circle action S^1 on $\mathrm{HH}(A/k)$ which after passing to homology induces an action

$$H_*(S^1, k) \cong k[b]/(b^2) \curvearrowright \mathrm{HH}_*(A/k)$$

which is precisely acting by the differential referenced above. Through HKR and this circle action, Hochschild homology is deeply related to both loop spaces and de Rham cohomology. As a sample theorem, one has the following:

Theorem 3.6 (Jones)**.** *Let X be a simply connected topological space. Then the singular cohomology of X is isomorphic to the Hochschild homology of the algebra of cochains $C^\bullet(X;k)$ on X with coefficients in k, i.e.*

$$H^*(X;k) \cong \mathrm{HH}_*(C^\bullet(X;k)).$$

4 Integrality issues and THH

To end these notes, we briefly mention some issues with the definition presented here and how more modern approaches correct these defects.

We saw in §2 that the functor we are interested in deriving is the pullback functor

$$\begin{aligned} \Delta^* : \mathrm{QC}(\operatorname{Spec} A \times_k \operatorname{Spec} A) &\longrightarrow \mathrm{QC}(\operatorname{Spec} A) \\ M &\longmapsto M \otimes_{A^e} A. \end{aligned}$$

Why might one do this? Well, in classical AG, at some point one realizes that rather than looking at the category of quasi-coherent sheaves $\mathrm{QC}(X)$ on X, one should instead work with the derived category $\mathcal{D}^b_{\mathrm{qc}}(X)$ of quasi-coherent sheaves on X. Making this jump, we see that pullback becomes

$$\begin{aligned} \Delta^* : \mathcal{D}^b_{\mathrm{qc}}(\operatorname{Spec} A \times_k \operatorname{Spec} A) &\longrightarrow \mathcal{D}^b_{\mathrm{qc}}(\operatorname{Spec} A) \\ M &\longmapsto M \otimes^{\mathbf{L}}_{A^e} A, \end{aligned}$$

i.e. the functor we consider here.

However, as one increases the level of sophistication, it becomes apparent that $\operatorname{Spec} A \times_k \operatorname{Spec} A$ should be interpreted as a *derived* fiber product, i.e. we should be doing DAG. In this case, pullback becomes

$$\begin{aligned} \Delta^* : \mathrm{QC}(\operatorname{Spec} A \times_k \operatorname{Spec} A) &\longrightarrow \mathrm{QC}(\operatorname{Spec} A) \\ M &\longmapsto M \otimes^{\mathbf{L}}_{A \otimes^{\mathbf{L}}_k A} A, \end{aligned} \tag{‡}$$

where now the tensor product defining A^e also becomes derived. This is sometimes called *Shukla homology*.

Finally, one may decide they want to work with SAG. In this case pullback along the diagonal still takes the form (‡) but the tensor products are now understood to be tensor products of modules over $\mathbb{E}_\infty$-rings. When one takes $M = A$ and the base $k = \mathbb{S}$ to be the sphere spectrum, this is typically referred to as *topological Hochschild homology*, denoted THH.

Proposition 4.1. *We have the following computations:*

(i) $\mathrm{HH}_*^{\mathrm{AG}}(\mathbb{F}_p/\mathbb{Z}) = \mathbb{F}_p$

(ii) $\mathrm{HH}_*(\mathbb{F}_p/\mathbb{Z}) = \mathbb{F}_p$

(iii) $\mathrm{HH}_*^{\mathrm{DAG}}(\mathbb{F}_p/\mathbb{Z}) = \mathbb{F}_p\langle x\rangle$*, the free divided powers algebra on a single generator of degree 2*

(iv) (Bökstedt) $\mathrm{HH}_*^{\mathrm{SAG}}(\mathbb{F}_p/\mathbb{S}) = \mathbb{F}_p[x]$*, the free polynomial algebra on a single generator of degree 2.*