

Geometrization: Primer for F -Gauges

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September 18, 2025

1 Introduction and motivation

Prismatic cohomology was introduced by Bhatt and Scholze [BS22] as a unifying framework for studying various cohomology theories associated to p -adic formal schemes. It has specializations to crystalline, de Rham, syntomic and étale cohomology. As explained in previous talks, prismatic cohomology is built by forming the *prismatic site*. That is, given a prism (A, I) and an $\bar{A} = A/I$ -algebra R , the prismatic site $(R/A)_{\Delta}$ is built by considering all prisms (B, IB) under (A, I) with reduction under R . This site is given the discrete topology, and the (derived) global sections of the sheaf $\mathcal{O}_{\Delta} : (B, IB) \mapsto B$ computes the prismatic cohomology of R over (A, I) .

For general topoi theoretic reasons, any $\mathcal{O}_{\Delta}$ -module on $(R/A)_{\Delta}$ has an associated (derived) global sections, allowing us to consider prismatic cohomology with coefficients in any sheaf on the prismatic site. However, if we wish to still be able to specialise prismatic cohomology with general coefficients to de Rham, étale and crystalline cohomology, then we must equip our sheaves with additional data. Bhatt and Lurie [BL22a, BL22c, BL22b] propose that the correct coefficients, roughly speaking, are in fact sheaves equipped with a filtration and Frobenius equivariance. They construct the appropriate category of coefficients by *geometrization*, i.e. by constructing a (derived) stack X^{Syn} such that $\text{QCoh}(X^{\text{Syn}})$ (or at least its compact objects) are the coefficients for prismatic cohomology on X .

Since prismatic cohomology may be viewed as a deformation of de Rham cohomology, it is useful to recall some more classical instances of geometrization, which is the goal of this talk. This talk is based on [BL22c, §2.2, §2.3].

2 Geometrization of filtrations and grading

Everything from here on out will be assumed to be appropriately derived. Fix a commutative ring R .

2.1 Graded modules

Definition 2.1. Let $\mathbb{Z}_{\text{disc}}$ denote the set $\mathbb{Z}$ viewed as a discrete category. The ∞ -category of graded R -modules is $\text{Fun}(\mathbb{Z}_{\text{disc}}, \text{Mod}_R)$.

The category of graded R -modules is symmetric monoidal via Day convolution

$$(F \otimes G)(n) = \text{colim}_{i+j=n} F(i) \otimes_R F(j) = \bigoplus_{i+j=n} F(i) \otimes_R F(j).$$

One should think of $F : \mathbb{Z}_{\text{disc}} \rightarrow \text{Mod}_R$ as remembering only the graded pieces, i.e. $F(n)$ represents the n -th graded component. There is a forgetful functor

$$\begin{aligned} \text{colim} : \text{Fun}(\mathbb{Z}_{\text{disc}}, \text{Mod}_R) &\longrightarrow \text{Mod}_R \\ F &\longmapsto \bigoplus_{n \in \mathbb{Z}} F(n) \end{aligned}$$

given by taking colimits, i.e. summing the graded pieces and forgetting the decomposition.

Theorem 2.2. *There is a canonical equivalence*

$$\text{Fun}(\mathbb{Z}_{\text{disc}}, \text{Mod}_R) \simeq \text{QCoh}(B\mathbb{G}_m)$$

under which the forgetful functor $\text{colim} : \text{Fun}(\mathbb{Z}_{\text{disc}}, \text{Mod}_R) \rightarrow \text{Mod}_R$ gets identified with pullback along $\pi : \text{Spec } R \rightarrow B\mathbb{G}_m$.

Proof (Sketch). $B\mathbb{G}_m$ classifies line bundles on R -schemes, hence it comes with a tautological bundle $\mathcal{O}(1)$ whose tensor powers we denote by $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$. The explicit equivalence is given in one direction by

$$\begin{aligned} \text{Fun}(\mathbb{Z}_{\text{disc}}, \text{Mod}_R) &\longrightarrow \text{QCoh}(B\mathbb{G}_m) \\ F &\longmapsto \bigoplus_{n \in \mathbb{Z}} F(n) \otimes_R \mathcal{O}(-n). \end{aligned}$$

□

We outline now two ways to think about this theorem. Whenever a group scheme G acts on a scheme X , we get that G also acts on $\text{QCoh}(X)$ where an element $g \in G$ acts by

$$T_g^* : \text{QCoh}(X) \rightarrow \text{QCoh}(X).$$

In a sufficiently nice setting, the projection map $X \rightarrow X/G$ is an fpqc cover, and thus $\text{QCoh}(X/G)$ may be computed from $\text{QCoh}(X)$ via descent. Doing this, one finds that

$$\text{QCoh}(X/G) \simeq \text{QCoh}(X)^{hG}$$

is the *homotopy* fixed points of the action $G \curvearrowright \mathrm{QCoh}(X)$. Informally, a homotopy fix point is a sheaf $\mathcal{F} \in \mathrm{QCoh}(X)$ along with an *equivalence* (not equality!)

$$\alpha_g : \mathcal{F} \simeq T_g^* \mathcal{F}$$

for every g such that $\alpha_{g_1 g_2} \simeq T_{g_2}^* \alpha_{g_1} \circ \alpha_{g_2}$ (ignoring higher equivalences and the topology on G).

In our setting, $X = \mathrm{Spec} R$ is a point and $\mathbb{G}_m$ is acting trivially, so we have that $T_g^* = \mathrm{id}$ for every $g \in \mathbb{G}_m$. Since we are taking homotopy fixed points, a homotopy fix point requires giving an R -module $M \in \mathrm{Mod}_R = \mathrm{QCoh}(\mathrm{Spec} R)$ along with an R -linear automorphism $\alpha_g : M \rightarrow M$ for every g such that $\alpha_{g_1 g_2} \simeq \alpha_{g_1} \alpha_{g_2}$, i.e. it is the data of a lift of M to a $\mathbb{G}_m$ -representation. That is to say,

$$\mathrm{QCoh}(B\mathbb{G}_m) \simeq \mathrm{Rep}(\mathbb{G}_m).$$

Thus the claim reduces to saying that a $\mathbb{G}_m$ -representation is the same as a graded R -module.

Since both $\mathbb{G}_m$ and $\mathrm{Spec} R$ are affine, by passing to functions one may show that a $\mathbb{G}_m$ -representation on an R -module M is the same as turning M into a comodule for the coalgebra $\mathcal{O}(\mathbb{G}_m) = R[t, t^{-1}]$. However, giving a coaction map

$$a : M \rightarrow M \otimes_R R[t, t^{-1}]$$

we obtain a grading on M by letting the n -th graded piece be the elements of M sent to pure tensors of the form $m \otimes t^n$. Conversely given a $\mathbb{Z}$ -grading on M , we get a coaction map by

$$\begin{aligned} a : M &\longrightarrow M \otimes_R R[t, t^{-1}] \\ m &\longmapsto \sum_{n \in \mathbb{Z}} m_n \otimes t^n \end{aligned}$$

where m_n are the homogeneous components of M .

Under this equivalence, pullback along $\pi : \mathrm{Spec} R \rightarrow B\mathbb{G}_m$ forgets the comodule structure and returns the underlying module M as claimed.

Remark 2.3. For the differential geometrically inclined, one may view that $\mathbb{G}_m$ as S^1 which is a torus. Thus any representation of S^1 will be able to be split into one dimensional irreducible representations. This gives a decomposition of our representation into representations indexed by the characters of S^1 . But the characters of S^1 , i.e. maps $S^1 \rightarrow \mathbb{C}^\times$, are determined by their degree. Thus we get a $\mathbb{Z}$ -indexed decomposition $V = \bigoplus_{n \in \mathbb{Z}} V_n$ where $\alpha \in S^1$ acts by multiplication by α^n on V_n .

2.2 Filtrations

Definition 2.4. The ∞ -category of decreasing filtered R -modules is given by $\mathcal{F}(R) = \text{Fun}(\mathbb{Z}^{\text{op}}, \text{Mod}_R)$ where $\mathbb{Z}$ is given its usual poset ordering.

Explicitly, a filtered R -module is given by a diagram of the form

$$\dots \longrightarrow F^{i+1} \longrightarrow F^i \longrightarrow F^{i-1} \longrightarrow \dots$$

where no assumptions are made on the transition maps, though to square with classical intuition you should image they are inclusions and that as i gets smaller the i -th filtered component gets larger.

Every filtered module F has an underlying module given by taking colimits

$$\text{colim} : \text{Fun}(\mathbb{Z}^{\text{op}}, \text{Mod}_R) \longrightarrow \text{Mod}_R$$

which we denote by $\underline{F}$. It also has an associated graded $\text{gr } \underline{F} \in \text{Fun}(\mathbb{Z}_{\text{disc}}, \text{Mod}_R)$ given by

$$\text{gr } \underline{F}(n) = \text{cofib}(F^{i+1} \rightarrow F^i).$$

We say that $\underline{F}$ is *complete* if $\lim_i F^i = 0$.

$\mathcal{F}(R)$ is symmetric monoidal via Day convolution, i.e. via

$$(F \otimes G)(n) = \text{colim}_{i+j \geq n} F(i) \otimes_R G(j).$$

Theorem 2.5. *Let $\mathbb{G}_m$ act on $\mathbb{A}^1 = \text{Spec } R[\lambda]$ by assigning λ to have weight 1. Then there is a canonical equivalence*

$$\text{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathcal{F}(R)$$

given in one direction by the Rees construction

$$\begin{aligned} \text{Rees} : \mathcal{F}(R) &\longrightarrow \text{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \\ F &\longmapsto \bigoplus_{n \in \mathbb{Z}} F(i) \cdot \lambda^{-i}. \end{aligned}$$

Under this identification and the identification of Theorem 2.2, pullback along $B\mathbb{G}_m = \text{pt}/\mathbb{G}_m \rightarrow \mathbb{A}^1/\mathbb{G}_m$ corresponds to taking the associated graded and pullback along $\text{pt} = \mathbb{G}_m/\mathbb{G}_m \rightarrow \mathbb{A}^1/\mathbb{G}_m$ corresponds to taking the underlying module.

In the Rees construction, multiplication by λ has transition maps given by the transition maps $F(i+1) \rightarrow F(i)$ in the filtration.

Morally, we may view this as a family of modules parametrized by a parameter $\lambda \in \mathbb{A}^1$. Rescaling the formal parameter λ leaves the construction invariant up to equivalence, so this family in fact is parametrized by $\mathbb{A}^1/\mathbb{G}_m$. Over the fiber $\lambda = 1$, we simply recover the underlying module. Setting $\lambda = 0$, we get

$$\begin{aligned} \text{Rees}(F) \otimes_R R[\lambda]/(\lambda) &\simeq \text{cofib}(\text{Rees}(F) \xrightarrow{\cdot \lambda} \text{Rees}(F)) \\ &\simeq \bigoplus_{n \in \mathbb{Z}} \text{cofib}(F(i+1) \cdot \lambda^{-i-1} \xrightarrow{\cdot \lambda} F(i) \cdot \lambda^{-i}) \\ &\simeq \bigoplus_{n \in \mathbb{Z}} \text{cofib}(F(i+1) \rightarrow F(i)) \\ &\simeq \text{gr } F \end{aligned}$$

as claimed.

Moreover, vanishing of the parameter λ corresponds to a Cartier divisor in $\mathbb{A}^1/\mathbb{G}_m$ whose associated line bundle we denote by $\mathcal{O}(-1)$, so the Rees construction may also be viewed as

$$F \longmapsto \bigoplus_{n \in \mathbb{Z}} F(i) \otimes \mathcal{O}(i).$$

3 Geometrization of de Rham cohomology

Our next goal is to geometrize the de Rham cohomology of a scheme X/k where k is of characteristic zero. Recall that for a k -scheme X we have the de Rham complex

$$\Omega_{X/k}^\bullet = \left(0 \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega_{X/k}^1 \xrightarrow{d} \Omega_{X/k}^2 \xrightarrow{d} \cdots \right)$$

whose (hyper)cohomology computes the de Rham cohomology of X . Our goal is to find a stack X_{dR} such that we have a canonical equivalence

$$R\Gamma(X_{\text{dR}}, \mathcal{O}_{X_{\text{dR}}}) \simeq R\Gamma(X, \Omega_{X/k}^\bullet).$$

One may in fact view this as an instance of a more general phenomenon: Given a vector bundle $\mathcal{E}$ on X with flat connection ∇ , we may extend the connection by the Leibniz rule to obtain a complex

$$0 \longrightarrow \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/k}^1 \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/k}^2 \longrightarrow \cdots$$

whose (hyper)cohomology provides us with a cohomology theory of vector bundles with flat connection. It turns out that X_{dR} geometrizes this whole phenomenon as vector bundles on X_{dR} are

equivalent to vector bundles on X with flat connection.

Recall as well that $\Omega_{X/k}^\bullet$ has a filtration given by forgetting terms in the chain complex, called the *Hodge filtration*, whose associated graded is the *Hodge complex* of X

$$\Omega_{X/k}^H = \left(0 \longrightarrow \mathcal{O}_X \xrightarrow{\circ} \Omega_{X/k}^1 \xrightarrow{\circ} \Omega_{X/k}^2 \xrightarrow{\circ} \cdots \right) = \bigoplus_i \Omega_{X/k}^i[-i].$$

Definition 3.1. Given a k -scheme X , we define the *de Rham stack* X_{dR} as a functor of points on finitely generated k -algebras by

$$X_{\text{dR}}(R) = X(R_{\text{red}})$$

where $R_{\text{red}} = R/\text{Nil } R$.

Morally, X_{dR} behaves like X , but is not able to distinguish between infinitesimally close points so that all maps $\text{Spec } R \rightarrow X_{\text{dR}}$ must factor through $\text{Spec } R_{\text{red}} \rightarrow X_{\text{dR}}$. We have the following theorem due to Simpson.

Theorem 3.2 (Simpson). *For a smooth k -scheme X , there is a natural identification*

$$R\Gamma(X_{\text{dR}}, \mathcal{O}_{X_{\text{dR}}}) \simeq R\Gamma(X, \Omega_{X/k}^\bullet).$$

Rather than prove this, our goal will be to explain how to upgrade this to keep track of the Hodge filtration on de Rham cohomology. Indeed, derived global sections is given by (derived) pushforward to a point. Thus if we want to remember the Hodge filtration, we should construct a variant X_{dR}^+ of X_{dR} which lives over $\mathbb{A}^1/\mathbb{G}_m$, then the (derived) pushforward of the structure sheaf along

$$\pi : X_{\text{dR}}^+ \rightarrow \mathbb{A}^1/\mathbb{G}_m$$

should recover the de Rham complex with its Hodge filtration.

Forgetting the filtration should correspond to taking the fibre over $\text{pt} = \mathbb{G}_m/\mathbb{G}_m \subseteq \mathbb{A}^1/\mathbb{G}_m$ and should recover the plain de Rham space X_{dR} , i.e. we should have a pullback

$$\begin{array}{ccc} X_{\text{dR}} & \longrightarrow & \mathbb{G}_m/\mathbb{G}_m \\ \downarrow & & \downarrow \\ X_{\text{dR}}^+ & \longrightarrow & \mathbb{A}^1/\mathbb{G}_m. \end{array}$$

We can also optionally only remember the associated graded portion by forming the fiber over $B\mathbb{G}_m \subseteq \mathbb{A}^1/\mathbb{G}_m$ which should give a space X_{Hodge} .

References

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