Some Notes on Category Theory

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1 Existence of Adjoints

It is a basic fact that every right adjoint preserves all limits, but when is the converse true? Given a continuous functor $F : \mathcal{C} \to \mathcal{D}$, when can we guarantee the existence of a left adjoint? The goal of this section is discuss some of the major theorems in category theory which guarantee existence of adjoints.

1.1 The General Adjoint Functor Theorem

Suppose we have a functor $F : \mathcal{C} \to \mathcal{D}$. If $G : \mathcal{D} \to \mathcal{C}$ is to be a left adjoint, we must have that

$$\mathrm{Hom}(Ga, b) \cong \mathrm{Hom}(a, Fb)$$

for every $a \in \mathcal{D}$ and $b \in \mathcal{C}$. In particular, Ga must represent the functor $\mathrm{Hom}(a, F-) : \mathcal{C} \to \mathrm{Set}$. In fact, this pointwise consideration is sufficient.

Proposition 1.1.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor such that* $\mathrm{Hom}(a, F-) : \mathcal{C} \to \mathrm{Set}$ *is representable for every $a \in \mathcal{D}$, then F has a left adjoint.*

Proof. If G is to be the left adjoint, then by the above discussion we are forced to set Ga to be a representing object for the functor $\mathrm{Hom}(a, F-)$. We now need to upgrade this to a functor.

For each $a \in \mathcal{D}$, fix a representing object Ga induced by $\eta_a \in \mathrm{Hom}(a, FGa)$. Now, given $f : a \to a'$, we get an induced natural transformation

$$\mathrm{Hom}(a, F-) \Longrightarrow \mathrm{Hom}(a', F-)$$

and thus an induced morphism $Ga \to Ga'$ which we define to be Gf. One then checks that $G : \mathcal{D} \to \mathcal{C}$ is a functor which is left adjoint to F by construction, and here $\eta : 1 \to FG$ ends up being the unit of this adjunction. □

We now digress slightly to discuss an alternate characterization of when a functor $L : \mathcal{C} \to \mathrm{Set}$ is representable that will be useful to us.

Definition 1.1.2. Let $F : \mathcal{C} \to \mathrm{Set}$ be a covariant functor. The *category of elements of F*, denoted $\int F$, is the category whose objects are pairs (c, x) where $c \in \mathcal{C}$ and $x \in Fc$ and whose morphisms $(c, x) \to (d, y)$ are morphisms $f : c \to d$ in $\mathcal{C}$ such that $(Ff)(x) = y$.

For $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ a contravariant functor, we define $\int F$ to be pairs (c, x) with $c \in \mathcal{C}, x \in Fc$ and morphisms $(c, x) \to (d, y)$ being maps $f : c \to d$ in $\mathcal{C}$ such that $(Ff)(y) = x$.

Example 1.1.3. We have that for a category $\mathcal{C}$ and $c \in \mathcal{C}$, $\int \mathrm{Hom}_{\mathcal{C}}(c, -)$ is the slice category $\mathcal{C}_{c/}$ and $\int \mathrm{Hom}_{\mathcal{C}}(-, c)$ is the slice category $\mathcal{C}_{/c}$.

Proposition 1.1.4. *A covariant functor* $F : \mathcal{C} \to \mathrm{Set}$ *is representable if and only if* $\int F$ *has an initial object. Dually, a contravariant functor is representable if and only if* $\int F$ *has a terminal object.*

Proof. Let $(c, x) \in \int F$. Then x induces a natural transformation $\eta : \mathrm{Hom}(c, -) \Rightarrow F$. Now, (c, x) is initial if and only if for every $d \in \mathcal{C}$ and $y \in Fd$, there exists a unique morphism $f : c \to d$ with $(Ff)(x) = y$. This, however, is exactly the statement that the induced map $\eta : \mathrm{Hom}(c, d) \to Fd$ is a bijection for all $d \in \mathcal{C}$, i.e. that (c, x) represents F. □

Corollary 1.1.5. *A functor* $F : \mathcal{C} \to \mathcal{D}$ *has a left adjoint if and only if for every* $d \in \mathcal{D}$*, the category* $d{\downarrow}F$ *has an initial object.*

Proof. This is a direct consequence of Proposition 1.1.1 and Proposition 1.1.4 upon realizing that $\int \mathrm{Hom}(d, F-) \cong d{\downarrow}F$. □

Thus we have reduced the question of having a left adjoint to that of determining whether each category $d{\downarrow}F$ has an initial object. For this we have the following proposition.

Proposition 1.1.6. *For any functor* $F : \mathcal{C} \to \mathcal{D}$ *and* $d \in \mathcal{D}$*, the forgetful functor* $\Pi : d{\downarrow}F \to \mathcal{C}$ *strictly creates any limits which exist in* $\mathcal{C}$ *and are preserved by* F*.*

In particular, if F *is continuous and* $\mathcal{C}$ *is complete, then* $d{\downarrow}F$ *is complete.*

Proof. Consider a diagram $D : J \to d{\downarrow}F$ and suppose that $c = \lim_J \Pi D$ exists and is preserved by F so that $Fc = \lim_J F\Pi D$. Now, the original diagram D provides a cone with summit d over $F\Pi D$, so we get a unique induced map $d \to Fc$. One then checks that $(c, d \to Fc) = \lim_J D$. □

This proposition will allow us to construct initial objects of $d{\downarrow}F$. Recall that initial objects, which are colimits of the empty diagram, can also be viewed as *limits* of the identity functor. One might then be tempted to conclude from Proposition 1.1.6 that if F is continuous and $\mathcal{C}$ is complete, then $d{\downarrow}F$ then has an initial object, but this is false—in general $d{\downarrow}F$ is large, so the diagram $\mathrm{id} : d{\downarrow}F \to d{\downarrow}F$ is not a small limit and thus not guaranteed by completeness.

The General Adjoint Functor Theorem comes from Proposition 1.1.6 by giving a sufficient condition for the limit $\mathrm{id} : d{\downarrow}F \to d{\downarrow}F$ to be computed via a small diagram.

Theorem 1.1.7 (General Adjoint Functor Theorem)**.** *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a continuous functor with* $\mathcal{C}$ *complete and locally small. Suppose additionally that for every* $d \in \mathcal{D}$ *the following* solution set condition *is satisfied:*

- *there exists a* set Φ_d *of morphisms* $d \to Fc_i$ *such that every morphism* $d \to Fc$ *factors through some* $d \to Fc_i \in \Phi_d$ *along a morphism* $c_i \to c$ *in* $\mathcal{C}$.

Then F *has a left adjoint.*

We will omit the proof, but the moral, as stated above, is that having the solution set condition allows us to replace the large limit for computing an initial object with a small limit.

Example 1.1.8. Consider the inclusion Haus $\hookrightarrow$ Top. This inclusion is continuous as the product and coequalizer of Hausdorff spaces is Hausdorff, which also shows that Haus is complete. Thus to show Haus $\hookrightarrow$ Top has a left adjoint, it suffices to demonstrate the solution set condition.

For this, fix a topological space X. Let $\mathcal{S}$ be a set of homeomorphism class representatives for all Hausdorff topological spaces with underlying set having cardinality $\leq |X|$. This is a set because there are only a sets worth of cardinals $\leq |X|$, and for each set of a given size, there's only a sets worth of topologies we can place on it. Now let Φ_X be the set of all continuous maps $X \to Y$ for some $Y \in \mathcal{S}$. Then given a map $f : X \to Z$ for any Hausdorff space Z, we have that $f(X)$ is a Hausdorff space of cardinality $\leq |X|$ and hence is homeomorphic to some element of $\mathcal{S}$. Hence we may factor f through an element of Φ_X.

It follows from the General Adjoint Functor Theorem that Haus $\hookrightarrow$ Top has a left adjoint, i.e. Haus is a reflective subcategory of Top. As a corollary, we learn that Haus is cocomplete as a reflective subcategory of the cocomplete category Top.

Example 1.1.9. In a similar manner to Exercise 1.1.8, one may show that the forgetful functor U : Group $\to$ Set has a left adjoint. This proves the existence of the free group functor.

1.2 The Special Adjoint Functor Theorem

The Special Adjoint Functor Theorem is another sufficient condition for reducing a large limit to a small limit so that we may apply Proposition 1.1.6.

To state the Special Adjoint Functor Theorem, we need the notion of subobjects.

Definition 1.2.1. Let $\mathcal{C}$ be a category and $c \in \mathcal{C}$. A *subobject* of c is a monomorphism $c' \hookrightarrow c$. We call two subobjects c', c'' isomorphic if there exists an isomorphism $c' \cong c''$ over c.

Definition 1.2.2. Given a collection of subobjects $c_i \hookrightarrow c$, the *intersection* of the c_i's, if it exists, is the limit of the diagram of the c_i's mapping into c.

Remark 1.2.3. It is easy to check that if $\bigcap_i c_i$ exists, then it is a suboject of each c_i and thus also of c.

We need one final definition before stating the theorem.

Definition 1.2.4. A *separating set* of a category $\mathcal{C}$ is a set Φ of objects of $\mathcal{C}$ such that for every two parallel morphisms $f, g : x \rightrightarrows y$ with $f \neq g$, there exists a morphism $h : c \longrightarrow x$ with $c \in \Phi$ such that $fh \neq gh$.

Dually, we have the notion of a coseparating set.

Example 1.2.5. By Urysohn's lemma, $\Phi = \{[0,1]\}$ is a coseparating set for cHaus. Indeed, given two distinct morphisms $f, g : X \longrightarrow Y$ between compact Hausdorff spaces, there exists some $x \in X$ such that $f(x) \neq g(x)$. Then by Urysohn's lemma, there exists a map $h : Y \longrightarrow I$ which separates $f(x)$ from $g(x)$ so that $hf \neq hg$.

Theorem 1.2.6 (Special Adjoint Functor Theorem)**.** *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a continuous functor with $\mathcal{C}$ complete and both $\mathcal{C}$ and $\mathcal{D}$ locally small. If $\mathcal{C}$ admits a coseparating set and any collection of subobjects of a fixed object in $\mathcal{C}$ admits an intersection, then F has a left adjoint.*

Proof. See [10, Theorem 4.6.10]. □

Example 1.2.7. Consider the forgetful functor $U : \text{cHaus} \longrightarrow \text{Top}$. By Example 1.2.5, we know that cHaus has a coseparating set and intersections of subobjects here corresponds to the usual intersection of subspaces and hence exist. Finally, cHaus is complete by Tychonoff's theorem and the fact that equalizers of compact Hausdorff spaces are compact Hausdorff, which also shows that U is continuous. Thus by the Special Adjoint Functor Theorem, U has a left adjoint which proves the existence of the Stone–Čech compactification $\beta : \text{Top} \longrightarrow \text{cHaus}$.

In fact, abstraction of the proof of the existence of Stone–Čech compactification is what gave birth to the Special Adjoint Functor Theorem.

Remark 1.2.8. Any category $\mathcal{C}$ which satisfies the hypotheses of Theorem 1.2.6 is also necessarily cocomplete. Indeed, for any small category J, the category $\mathcal{C}^J$ is also locally small and the constant diagram functor $\Delta : \mathcal{C} \longrightarrow \mathcal{C}^J$ is continuous. Thus it has a left adjoint by the Special Adjoint Functor Theorem which implies that $\mathcal{C}$ has all colimits of shape J.

1.3 Freyd's Representability Theorem

Let $F : \mathcal{C} \longrightarrow \text{Set}$ be a continuous functor with $\mathcal{C}$ complete. The solution set condition of Theorem 1.1.7 is there to provide an initial object to $S \downarrow F$ for every set F. However, if we only care about F being representable, then we need only construct an initial object of $* \downarrow F \cong \int F$. This is the content of Freyd's Representability Theorem.

Theorem 1.3.1 (Freyd's Representability Theorem)**.** *Let $F : \mathcal{C} \longrightarrow \text{Set}$ be a continuous functor with $\mathcal{C}$ complete and locally small. Suppose that F satisfies the following solution set condition:*

- *there exists a set Φ of objects of $\mathcal{C}$ such that for every $c \in \mathcal{C}$ and $x \in Fc$, there exists some $s \in \Phi$, $y \in Fs$ and $f : s \to c$ such that $(Ff)(y) = x$.*

Then F is representable.

Proof. The solution set guarantees the existence of an initial object of $* \downarrow F \cong \int F$. □

1.4 Consequences for presentable categories

These adjoint functor theorems have an appealing consequences for a certain class of categories which we introduce now.

Definition 1.4.1. Let κ be a cardinal. A *κ-compact object* of $\mathcal{C}$ is an object $c \in \mathcal{C}$ such that $\mathrm{Hom}(-, c)$ commutes with κ-filtered colimits.

Definition 1.4.2. A category $\mathcal{C}$ is *κ-accessible* if it has all κ-filtered colimits and there exists a *set* of κ-compact objects which generate $\mathcal{C}$ under κ-filtered colimits. We say that $\mathcal{C}$ is *accessible* if it is κ-accessible for some κ.

A functor $F : \mathcal{C} \to \mathcal{D}$ is called *κ-accessible* if F commutes with κ-filtered colimits. We say that F is accessible if it is κ-accessible for some κ.

Definition 1.4.3. A locally small category $\mathcal{C}$ is called *presentable* (resp. κ-presentable) if $\mathcal{C}$ is accessible (resp. κ-accessible) and cocomplete.

Remark 1.4.4. If $\mathcal{C}$ is κ-presentable, then $\mathcal{C}$ is λ-presentable for all $\lambda \geq \kappa$. Similarly for accessibility.

Theorem 1.4.5. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between two presentable categories. Then F*

(i) admits a right adjoint if and only if it is cocontinuous

(ii) admits a left adjoint if and only if it is continuous and accessible

Proof. (i) follows from the Special Adjoint Functor Theorem and (ii) follows from the General Adjoint Functor Theorem. See [10, Theorem 4.6.17] for further references. □

1.4.1 Gabber's result

We now discuss an application of this to classical algebraic geometry.

Definition 1.4.6. Let κ be a cardinal. We say that an $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is *κ-generated* if there exists an open cover $X = \bigcup_i U_i$ such that each $\mathcal{F}|_{U_i}$ is generated by at most κ global sections $\mathcal{F}(U_i)$.

Remark 1.4.7. Any κ-generated $\mathcal{O}_X$-module is κ-compact in $\mathcal{O}_X-\mathrm{Mod}$. Moreover, since $\mathrm{QCoh}(X) \hookrightarrow \mathcal{O}_X-\mathrm{Mod}$ is cocontinuous, if $\mathcal{F} \in \mathrm{QCoh}(X)$ is κ-generated as an $\mathcal{O}_X$-module, then it is κ-compact in $\mathrm{QCoh}(X)$.

Proposition 1.4.8 (Gabber). *For any scheme X, there exists a cardinal κ such that for any $\mathcal{F} \in \mathrm{QCoh}(X)$, $\mathcal{F}$ is the direct limit of its κ-generated subsheaves.*

We first need a lemma.

Lemma 1.4.9. *Let X be a scheme and κ a cardinal. There are only a sets worth of isomorphism classes of κ-generated $\mathcal{O}_X$-modules.*

Proof. First, note that there are only a sets worth of quotients $\bigoplus_\kappa \mathcal{O}_X \twoheadrightarrow \mathcal{F}$. Now, a κ-generated $\mathcal{O}_X$-module is locally the quotient of a κ direct sum of structure sheaves, so the result follows from the fact that there is only a sets worth of open covers of X and then a sets worth of gluing data upon specifying the restriction of $\mathcal{F}$. □

Proof of Proposition 1.4.8 (Sketch). Our goal is to show that any quasi-coherent sheaf $\mathcal{F}$ is generated as a (directed) colimit of its κ-generated subsheaves for κ sufficiently large. The result then follows from Lemma 1.4.9. If X were affine, then we could take $\kappa = \omega$ since any module is the directed colimit of its finitely generated submodules. For a more general scheme X, we must make κ larger as we must include more generators in order to glue. For details see [1, Tag 077K]. □

Gabber's result (Proposition 1.4.8) nearly tells us that $\mathrm{QCoh}(X)$ is presentable. The issue, however, is that we have only shown that every object is a direct limit of a set of κ-compact objects, but not necessarily a κ-filtered colimit. Upgrading Gabber's result to the presentability of $\mathrm{QCoh}(X)$ takes some work.

Definition 1.4.10. An abelian category $\mathcal{C}$ is called a *Grothendieck abelian category* if

(i) small filtered colimits are exact

(ii) $\mathcal{C}$ is cocomplete

(iii) $\mathcal{C}$ admits a generator, i.e. an object A such that $\mathrm{Hom}_{\mathcal{C}}(A, -)$ is faithful.

Theorem 1.4.11 ([3, Proposition 3.10]). *Every Grothendieck abelian category is presentable.*

Remark 1.4.12. In fact, [3, Proposition 3.10] shows that in an abelian category where small filtered colimits are exact, $\mathcal{C}$ being Grothendieck abelian is *equivalent* to being presentable.

Grothendieck abelian categories also have other nice properties, such as there always exists sufficiently many injectives.

Theorem 1.4.13. *For any scheme X,* $\mathrm{QCoh}(X)$ *is Grothendieck abelian. In particular,* $\mathrm{QCoh}(X)$ *is presentable.*

Proof. By Gabber's result (Proposition 1.4.8), we have that every quasi-coherent sheaf $\mathcal{F}$ is the direct limit of its κ-generated subsheaves. Now, by Lemma 1.4.9 there are only a sets worth of isomorphism classes of κ-generated quasi-coherent sheaves. Let S be a set of representatives of these isomorphism classes. Then $\bigoplus_{\mathcal{G} \in S} \mathcal{G}$ is a generator for $\mathrm{QCoh}(X)$. □

Corollary 1.4.14. *The inclusion* $\mathrm{QCoh}(X) \hookrightarrow \mathcal{O}_X\text{–Mod}$ *has a right adjoint* $Q : \mathcal{O}_X\text{–Mod} \to \mathrm{QCoh}(X)$.

Proof. The inclusion $\mathrm{QCoh}(X) \hookrightarrow \mathcal{O}_X\text{–Mod}$ is colimit preserving. Since $\mathrm{QCoh}(X)$ is presentable by Theorem 1.4.13 and $\mathcal{O}_X\text{–Mod}$ is presentable (exercise!), we get by Theorem 1.4.5 that $\mathrm{QCoh}(X) \hookrightarrow \mathcal{O}_X\text{–Mod}$ has a right adjoint. □

This has some nice corollaries. For example, given a morphism $f : X \to Y$, we know that $f^* : \mathcal{O}_Y\text{–Mod} \rightleftarrows \mathcal{O}_X\text{–Mod} : f_*$ is an adjoint pair. As a consequence, if f_* preserves quasi-coherence, e.g. if f is proper or qcqs, then $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is right adjoint to $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$. However, in general, the right adjoint to $f^*|_{\mathrm{QCoh}(Y)}$ will be given by $Qf_*|_{\mathrm{QCoh}(X)}$ where $Q : \mathcal{O}_Y\text{–Mod} \to \mathrm{QCoh}(Y)$ is the adjoint given by Corollary 1.4.14.

2 Monads and Barr–Beck

2.1 Monoidal categories, algebras and modules

In this section, we discuss the general notion of a monoidal category as well their algebra objects and modules over said algebra objects.

The goal of monoidal categories is to abstract the notion of a "category with a tensor product" where we drop the condition that the tensor product is commutative in its two arguments. We are of the opinion that the proper way to define the coherence conditions of a monoidal category are ∞-categorical in nature, so we only sketch the definition for 1-categories.

Definition 2.1.1 (Sketch). A *monoidal category* is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object $1 \in \mathcal{C}$ such that there exists natural isomorphisms

$$a : ((- \otimes -) \otimes -) \xrightarrow{\simeq} (- \otimes (- \otimes -))$$

and

$$1 \otimes (-) \xrightarrow{\simeq} \mathrm{id}_{\mathcal{C}}$$
$$(-) \otimes 1 \xrightarrow{\simeq} \mathrm{id}_{\mathcal{C}}$$

which are compatible in a suitable way.

Given a monoidal category $(\mathcal{C}, \otimes, 1)$, we may consider the category $\mathrm{Alg}(\mathcal{C})$ of algebra objects in $\mathcal{C}$ defined as follows.

Definition 2.1.2. An algebra object in $(\mathcal{C}, \otimes, 1)$ is an object $A \in \mathcal{C}$ together with a multiplication map $m : A \otimes A \to A$ and unit map $e : 1 \to A$ satisfying the following categorified associative and unital axioms:

(i) the diagram

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[r, "m\otimes\mathrm{id}"] \arrow[d, "\cong"] & A \otimes A \arrow[dd, "m"] \\
A \otimes (A \otimes A) \arrow[d, "\mathrm{id}\otimes m"] & \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

commutes

(ii) the diagrams

$$\begin{tikzcd}
A \cong 1 \otimes A \arrow[r, "e\otimes\mathrm{id}"] \arrow[rr, bend right, "\mathrm{id}"'] & A \otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \cong A \otimes 1 \arrow[r, "\mathrm{id}\otimes e"] \arrow[rr, bend right, "\mathrm{id}"'] & A \otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

commute.

A *homomorphism* between algebra objects $(A, m, e) \to (A', m', e')$ is a morphism $f : A \to A'$ such that

$$\begin{tikzcd}
A \otimes A \arrow[r, "m"] \arrow[d, "f\otimes f"] & A \arrow[d, "f"] \\
A' \otimes A' \arrow[r, "m'"] & A'
\end{tikzcd}$$

commutes. The category of algebra objects and homomorphisms is denoted by $\mathrm{Alg}(\mathcal{C})$.

Remark 2.1.3. The category Cat of all (small) categories is a monoidal category via the cartesian product ×. Then, a monoidal category is simply an algebra object of $(\mathrm{Cat}, \times)$.

Finally, given a monoidal category $\mathcal{C}$ and $A \in \mathrm{Alg}(\mathcal{C})$, we wish to discuss modules over A. For this to make sense, we need a category on which $\mathcal{C}$ acts on in which our objects will live.

Definition 2.1.4 (Sketch)**.** Let $(\mathcal{C}, \otimes)$ be a monoidal category and $\mathcal{D}$ another category. We say that $\mathcal{D}$ is *tensored* over $\mathcal{C}$ if for any $A \in \mathcal{C}$ and $X \in \mathcal{D}$, there exists a well defined object $A \otimes X$ which is natural in A and X and satisfies appropriate unital and associative axioms.

Definition 2.1.5. Let $\mathcal{D}$ be tensored over $(\mathcal{C}, \otimes)$ and $(A, m, e) \in \mathrm{Alg}(\mathcal{C})$. A *module over A in $\mathcal{D}$* is an object $M \in \mathcal{D}$ along with an action map $a : A \otimes M \to A$ such that

(i) the diagram

$$\begin{tikzcd}
A \otimes (A \otimes M) \cong (A \otimes A) \otimes M \arrow[r, "m\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes a"] & A \otimes M \arrow[d, "a"] \\
A \otimes M \arrow[r, "a"] & M
\end{tikzcd}$$

commutes

(ii) the diagram

$$\begin{tikzcd}
M \cong 1 \otimes M \arrow[r, "e\otimes\mathrm{id}"] \arrow[rr, bend right, "\mathrm{id}"] & A \otimes M \arrow[r, "a"] & M
\end{tikzcd}$$

commutes.

A *homomorphism* between two modules $f : (M, a) \to (M', a')$ is given by a morphism $f : M \to M'$ such that

$$\begin{tikzcd}
A \otimes M \arrow[r, "\mathrm{id}\otimes f"] \arrow[d, "a"] & A \otimes M' \arrow[d, "a'"] \\
M \arrow[r, "f"] & M'
\end{tikzcd}$$

commutes. The category of A-modules in $\mathcal{C}$ is denoted by $\mathrm{Mod}_A(\mathcal{C})$.

Given any algebra object A, we always have a forgetful functor $U : \mathrm{Mod}_A(\mathcal{C}) \to \mathcal{C}$ which forgets the module structure. This functor also always has a left adjoint given by the *free module* functor $F : \mathcal{C} \to \mathrm{Mod}_A(\mathcal{C})$ which sends an object $S \in \mathcal{C}$ to the module $A \otimes S$ with scalar multiplication given by

$$A \otimes (A \otimes S) \cong (A \otimes A) \otimes S \xrightarrow{m\otimes\mathrm{id}} A \otimes S.$$

2.2 Monads

Fix a category $\mathcal{C}$. We may turn $\mathrm{End}(\mathcal{C})$ into a strictly monoidal category via composition of endofunctors, and we have that $\mathcal{C}$ is tensored over $\mathrm{End}(\mathcal{C})$ via evaluation of endofunctors.

Definition 2.2.1. A *monad* acting on a category $\mathcal{C}$ is an algebra object of $\mathrm{End}(\mathcal{C})$.

Proposition 2.2.2. *Given an adjunction $F \dashv U$, let $T = UF$, $\eta : 1 \Rightarrow T$ be the unit of adjunction and $\varepsilon : T \Rightarrow 1$ be the counit of adjunction. Then $(T, U\varepsilon F, \eta)$ is a monad.*

Proof. A tedious diagram chase. ☐

Two natural questions then arise:

- Does every monad arise in this way?
- Given an adjunction $F \dashv U$, $U : \mathcal{D} \to \mathcal{C}$, when can we recover $\mathcal{D}$ as $\mathrm{Mod}_T(\mathcal{C})$?

The answer to the first question is yes, and we should study the second question by analyzing all such categories and adjunctions which give rise to T.

Proposition 2.2.3. *Let T be a monad on $\mathcal{C}$. Then the monad on $\mathcal{C}$ arising from the free forgetful adjunction $F : \mathcal{C} \rightleftarrows \mathrm{Mod}_T(\mathcal{C}) : U$ is isomorphic to T.*

Corollary 2.2.4. *Let $\mathrm{Free}_T \subseteq \mathrm{Mod}_T(\mathcal{C})$ be the full subcategory of free T-modules, i.e. the essential image of $F : \mathcal{C} \to \mathrm{Mod}_T(\mathcal{C})$. Then the monad arising from the adjunction $F : \mathcal{C} \rightleftarrows \mathrm{Free}_T : U$ is isomorphic to T.*

Classically these categories have names.

Definition 2.2.5. The category $\mathrm{Mod}_T(\mathcal{C})$ is referred to as the Eilenberg–Moore category, and the full subcategory $\mathrm{Free}_T \subseteq \mathrm{Mod}_T(\mathcal{C})$ is referred to as the Kleisli category.

These two categories are in fact extremal solutions to the question of whether every monad arises from an adjunction. Consider the following category $\mathrm{Adj}(T)$: The objects are adjunctions $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ inducing the monad T, and the morphisms are commuting diagrams

$$\begin{tikzcd}
\mathcal{D} \arrow[rr] \arrow[dr, bend left, "U"] & & \mathcal{D}' \arrow[dl, bend right, "U'"'] \\
& \mathcal{C}. \arrow[ul, bend left, "F"] \arrow[ul, phantom, "\dashv" sloped] \arrow[ur, bend right, "F'"'] \arrow[ur, phantom, "\dashv" sloped] &
\end{tikzcd}$$

Theorem 2.2.6. $\mathrm{Mod}_T(\mathcal{C})$ *is terminal in* $\mathrm{Adj}(T)$ *and* Free_T *is initial in* $\mathrm{Adj}(T)$.

2.3 The monadicity theorem

We now return to our second question posed at the beginning of the previous section. Given an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ with induced monad T on $\mathcal{C}$, when can we recover $\mathcal{D}$ as $\mathrm{Mod}_T(\mathcal{C})$?

Theorem 2.2.6 tells us that, if $F^T : \mathcal{C} \rightleftarrows \mathrm{Mod}_T(\mathcal{C}) : U^T$ is the free forgetful adjunction, then there exists a unique morphism $\mathcal{D} \to \mathrm{Mod}_T(\mathcal{C})$ making

$$\begin{tikzcd}
\mathcal{D} \arrow[rr] \arrow[dr, bend left, "U"] & & \mathrm{Mod}_T(\mathcal{C}) \arrow[dl, bend right, "U^T"'] \\
& \mathcal{C}. \arrow[ul, bend left, "F"] \arrow[ur, bend right, "F^T"'] \arrow[ul, phantom, "\dashv"] \arrow[ur, phantom, "\dashv"'] &
\end{tikzcd}$$

commute. It thus suffices to analyze when this induced map is an equivalence. We call right adjoints $U : \mathcal{D} \to \mathcal{C}$ inducing an equivalence $\mathcal{D} \simeq \mathrm{Mod}_T(\mathcal{C})$ *monadic*.

To develop some obstructions to this, we need to study the adjunction $F^T : \mathcal{C} \rightleftarrows \mathrm{Mod}_T(\mathcal{C}) : U^T$. Consider the following diagram in $\mathrm{Mod}_T(\mathcal{C})$ for some T-module A:

$$\begin{tikzcd}
T \otimes T \otimes A \arrow[r, shift left, "m\otimes\mathrm{id}"] \arrow[r, shift right, "\mathrm{id}\otimes a"'] & T \otimes A \arrow[r, "a"] & A
\end{tikzcd} \tag{2.3.1}$$

One may check that this is a coequalizer diagram which is morally representing A via generators and relations. Moreover, after applying U^T, this diagram is in fact *split* in $\mathcal{C}$ via

$$\begin{tikzcd}
T \otimes T \otimes A \arrow[r, shift left, "m\otimes\mathrm{id}"] \arrow[r, shift right, "\mathrm{id}\otimes a"'] & T \otimes A \arrow[r, "a"] \arrow[l, bend left, "\mathrm{id}\otimes\eta_A"] & A \arrow[l, bend left, "\eta_A"]
\end{tikzcd}$$

where here $\eta : 1 \Rightarrow T$ is the unit of the monad T. Note, however, that η_A is not a homomorphism of algebras so this splitting does not hold in $\mathrm{Mod}_T(\mathcal{C})$. It turns out that every diagram indexed by $\bullet \rightrightarrows \bullet$ in $\mathrm{Mod}_T(\mathcal{C})$ which admits a split coequalizer in $\mathcal{C}$ after applying U^T has a lift to a coequalizer in $\mathrm{Mod}_T(\mathcal{C})$, as we will show now.

Remark 2.3.1. For those familiar, (2.3.1) is the start of the augmented bar complex. Since we are in the 1-categorical context, we only need to remember the first two terms when computing the colimit, but in an ∞-categorical context we should extend this complex into a full simplicial or semi-simplicial diagram. Further terms would represent relations between relations, i.e. syzygies, then relations between syzygies, and so on.

Definition 2.3.2. Let $\mathcal{C}$ be a category. A *split coequalizer* is a diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] \arrow[l, bend left, "s"] & C \arrow[l, bend left, "t"]
\end{tikzcd}$$

such that $fh = gh$, $ht = \mathrm{id}_C$, $gs = \mathrm{id}_B$ and $fs = th$.

Proposition 2.3.3. *The underlying fork of a split coequalizer diagram is a coequalizer which is preserved by any functor.*

Proof. Since any functor preserves split coequalizers, it suffices to show that the underlying fork of a split coequalizer is a coequalizer. For this, suppose we have a morphism $k : B \longrightarrow D$ such that $kf = kg$. Then

$$k = kgs = kfs = kth$$

so k factors through h. One easily checks that this factorization is unique as well, since h is a split epimorphism. $\square$

Definition 2.3.4. Let $U : \mathcal{D} \longrightarrow \mathcal{C}$. We say that U *creates U-split coequalizers* if for any diagram D of the form $\bullet \rightrightarrows \bullet$ in $\mathcal{D}$ which extends to a split coequalizer diagram in $\mathcal{C}$ under U, D extends to a coequalizer diagram in $\mathcal{D}$ which lifts via U the underlying coequalizer of this split coequalizer. Moreover, any fork in $\mathcal{D}$ which lifts the underlying coequalizer should be a coequalizer in $\mathcal{D}$.

We say that U *strictly* creates U-split coequalizers if U creates U-split coequalizers and the lift is unique.

Proposition 2.3.5. *The forgetful functor $U^T : \mathrm{Mod}_T(\mathcal{C}) \longrightarrow \mathcal{C}$ strictly creates U^T-split coequalizers.*

Proof. Suppose that we have a diagram

$$\begin{tikzcd}
M \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & N
\end{tikzcd}$$

of T-modules such that in $\mathcal{C}$ we have a split coequalizer diagram

$$\begin{tikzcd}
M \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & N \arrow[l, bend left=50, "s"] \arrow[r, "h"] & A. \arrow[l, bend left=50, "t"]
\end{tikzcd} \tag{2.3.2}$$

We need to give A the structure of a T-module, ideally in which h becomes a T-module homomorphism. Now, since split coequalizer are absolute, we have that the underlying fork of (2.3.2) is still a coequalizer after applying T. Thus we have a diagram

$$\begin{tikzcd}
T \otimes M \arrow[r, shift left, "\mathrm{id} \otimes f"] \arrow[r, shift right, "\mathrm{id} \otimes g"'] \arrow[d] & T \otimes N \arrow[r, "\mathrm{id} \otimes h"] \arrow[d] & T \otimes A \arrow[d, dashed, "\exists !"] \\
M \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & N \arrow[r, "h"] & A.
\end{tikzcd} \tag{2.3.3}$$

This unique filler map defines a morphism $a : T \otimes A \longrightarrow A$ which will be the action map for A. One then checks that this turns A into a T-module, and then commutativity of (2.3.3) implies that h is a homomorphism. Thus we have a lift of the underlying fork of (2.3.2). One checks that this is indeed a coequalizer in $\mathrm{Mod}_T(\mathcal{C})$, and in fact that this is the unique such lift. □

Theorem 2.3.6 (Barr–Beck)**.** *A right adjoint $U : \mathcal{D} \longrightarrow \mathcal{C}$ is monadic if and only if it creates U-split coequalizers.*

Proof. Only if follows from Proposition 2.3.5. For the other direction, let $K : \mathcal{D} \longrightarrow \mathrm{Mod}_T(\mathcal{C})$ be the canonical comparison functor and let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be the left adjoint of U. Then we have that $KF = F^T$ and $U^T K = U$. Thus if $L : \mathrm{Mod}_T(\mathcal{C}) \longrightarrow \mathcal{D}$ is a proposed inverse equivalence, we must have that $F \simeq LF^T$ and $U^T \simeq UL$.

We thus define L on free T-modules via

$$L(T \otimes A) = FA.$$

We now define L on a general T-module (M, a) such that

$$\begin{tikzcd} FUFU^TM = F(T \otimes U^TM) \arrow[r, shift left, "Fa"] \arrow[r, shift right, "\varepsilon_{FU^TM}"'] & FU^TM \arrow[r, dashed] & LM \end{tikzcd} \tag{2.3.4}$$

is a coequalizer, where $\varepsilon : FU \Rightarrow 1$ is the counit of the adjunction. The coequalizer in (2.3.4) exists because it is U-split and U creates U-split coequalizers. Indeed, under U, the diagram is

$$\begin{tikzcd} T \otimes T \otimes U^TM \arrow[r, shift left, "\mathrm{id} \otimes a"] \arrow[r, shift right, "m \otimes \mathrm{id}"'] & T \otimes U^TM \end{tikzcd}$$

which is extends to a split coequalizer diagram by the discussion at the beginning of this section.

Given a morphism $f : (A, a) \longrightarrow (B, b)$ of T-modules, we define Lf to be the unique dashed arrow making

$$\begin{tikzcd} FUFU^TA = F(T \otimes U^TA) \arrow[r, shift left, "Fa"] \arrow[r, shift right, "\varepsilon_{FU^TA}"'] \arrow[d, "FUFU^Tf"] & FU^TA \arrow[r] \arrow[d, "FU^Tf"] & LA \arrow[d, dashed, "Lf"] \\ FUFU^TB = F(T \otimes U^TB) \arrow[r, shift left, "Fa"] \arrow[r, shift right, "\varepsilon_{FU^TB}"'] & FU^TB \arrow[r] & LB \end{tikzcd}$$

commute.

One then checks that L defines an inverse equivalence to K. For details, see [10, Theorem 5.5.1]. □

There exists an alternate formulation of Theorem 2.3.6 which separates the reflection and lifting of U-split coequalizers into two parts.

Theorem 2.3.7. *A right adjoint $U : \mathcal{D} \to \mathcal{C}$ is monadic if and only if*

(i) every diagram $\bullet \rightrightarrows \bullet$ in $\mathcal{D}$ which under U extends to a split coequalizer in $\mathcal{C}$ has a colimit in $\mathcal{D}$ which is preserved by U

(ii) U is conservative

There also exists a frequently stated *sufficient* but not necessary condition for U to be monadic.

Definition 2.3.8. A pair of maps $f, g : A \to B$ is *reflexive* if there exists a common section $s : B \to A$, i.e. s such that $fs = gs = \mathrm{id}_B$.

Theorem 2.3.9 (Crude Monadicity Theorem)**.** *A right adjoint $U : \mathcal{D} \to \mathcal{C}$ is monadic if*

(i) $\mathcal{D}$ has and U preserves coequalizers of reflexive pairs

(ii) U is conservative

Example 2.3.10. Barr–Beck may used to simplify the proof of various descent theorems in algebraic geometry. Suppose that $f : U \to X$ is a faithfully flat, quasi-compact morphism of schemes. Then we have an adjoint pair $f^* : \mathrm{QCoh}(X) \rightleftarrows \mathrm{QCoh}(U) : f_*$ and we claim that f^* is comonadic. For this we verify the conditions of the crude monadicity theorem:

- f^* is conservative since f is faithfully flat.
- $\mathrm{QCoh}(X)$ is complete, so it has all equalizers. Moreover, f is flat so f^* preserves small limits.

Thus we get that f^* is comonadic so $\mathrm{QCoh}(X) \simeq \mathrm{CoMod}_{f^*f_*}(\mathrm{QCoh}(U))$. Thus it remains to show that $\mathrm{CoMod}_{f^*f_*}(\mathrm{QCoh}(U))$ is precisely descent data along f. For this, consider the pullback

$$\begin{tikzcd}
U \times_X U \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & U \arrow[d, "f"] \\
U \arrow[r, "f"] & X.
\end{tikzcd}$$

Let $T = f^*f_*$ be our comonad. By flat base change we have that $f^*f_* \simeq (\mathrm{pr}_1)_*(\mathrm{pr}_2)^*$ so $T = (\mathrm{pr}_1)_*(\mathrm{pr}_2)^*$. Thus the datum of a T-comodule is a $\mathcal{F} \in \mathrm{QCoh}(U)$ with a map $\mathcal{F} \to (\mathrm{pr}_1)_*(\mathrm{pr}_2)^*\mathcal{F}$, which is equivalently a map $(\mathrm{pr}_1)^*\mathcal{F} \to (\mathrm{pr}_2)^*\mathcal{F}$, subject to certain axioms. Finally, one checks that after translating the axioms for $\mathcal{F} \to T\mathcal{F}$ to define a T-comodule to $(\mathrm{pr}_1)^*\mathcal{F} \to (\mathrm{pr}_2)^*\mathcal{F}$ one gets the standard cocycle condition for descent data.

Example 2.3.11. The forgetful functor $U : \mathrm{cHaus} \to \mathrm{Set}$ is monadic. Firstly, it has a left adjoint given by Stone-Čech compactification. To show monadicity, we check the conditions of Theorem 2.3.7. Firstly, notice first that U is conservative since any bijection between compact Hausdorff spaces is a homeomorphism. This follows from the fact that every continuous map between compact Hausdorff spaces is automatically closed.

For the second condition, notice that an equivalent way to describe a topology on a set X is to given a closure operator $\overline{(-)} : \mathcal{P}(X) \to \mathcal{P}(X)$ with satisfies $A \subseteq \overline{A}$ and $\overline{\overline{A}} = A$. The corresponding topology is then the one in which closed sets are given by the image of $\overline{(-)}$, upon which $\overline{(-)}$ simply becomes the topological closure. A map $f : X \to Y$ between topological spaces is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for every $A \subseteq X$, and if X, Y are compact Hausdorff then every continuous map is already closed. Thus for a compact Hausdorff spaces, a continuous map is a set theoretic map $f : X \to Y$ which *commutes* with the closure operator, i.e. $f(\overline{A}) = \overline{f(A)}$. This in hand, suppose that X, Y are compact Hausdorff spaces and we have a split coequalizer

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] \arrow[l, bend left=50, "s"] & Z \arrow[l, bend left=50, "t"]
\end{tikzcd}$$

in Set. We may topologize Z by declaring that a subset $A \subseteq Z$ is closed if and only if $h^{-1}(A)$ is closed. Since split coequalizers are absolute, we still have a coequalizer after applying $\mathcal{P}$ and thus we may construct the dashed arrow below:

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\overline{(-)}"] & \mathcal{P}(Y) \arrow[r, "h"] \arrow[d, "\overline{(-)}"] & \mathcal{P}(Z) \arrow[d, dashed, "\overline{(-)}"] \\
\mathcal{P}(X) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathcal{P}(Y) \arrow[r, "h"] & \mathcal{P}(Z)
\end{tikzcd}$$

To see that this is a closure operator on $\mathcal{P}(Z)$, if $A \subseteq Z$ then

$$\overline{A} = \overline{h(t(A))} = h(\overline{t(A)}) \supseteq h(t(A)) = A$$

and

$$\overline{\overline{A}} = h(\overline{\overline{t(A)}}) = h(\overline{t(A)}) = \overline{h(t(A))} = \overline{A}.$$

From this, we see that Z can be turned into a topological space such that h is continuous and closed. It follows that Z is compact Hausdorff since it is the surjection of a compact Hausdorff space by a continuous, closed map. It then just remains to check that $X \rightrightarrows Y \to Z$ is a coequalizer in cHaus, which is not hard.

3 Model categories

3.1 Motivation and localization of categories

Suppose we have a category $\mathcal{C}$ and a class of morphisms W that we want to invert, a classical example being $\mathcal{C} = \mathrm{Top}$ and W weak equivalences between topological spaces. The category $\mathcal{C}[W^{-1}]$ formed by freely inverting the morphisms in W should satisfy a universal property: There should exist a localization map $i : \mathcal{C} \to \mathcal{C}[W^{-1}]$ sending morphisms to W to isomorphisms such that for any category $\mathcal{D}$, the functors $\mathcal{C}[W^{-1}] \to \mathcal{D}$ are in equivalence via pre-composition with i with functors $\mathcal{C} \to \mathcal{D}$ sending the morphisms in W to isomorphisms in $\mathcal{D}$.

For this localization to be reasonable, W should satisfy certain weak conditions, similar to how localization of rings in algebra require our localizing set to be closed under multiplication.

Definition 3.1.1. A *category with weak equivalences* $(\mathcal{C}, W)$ is a category $\mathcal{C}$ together with a collection of morphisms W containing all isomorphisms and satisfying the *two-out-of-three* property: Given a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

in $\mathcal{C}$, if any two of f, g, h belong to W, then so does the third.

Remark 3.1.2. We do not lose any generality by enforcing the two-out-of-three condition and that we contain all isomorphisms. Indeed, for any category $\mathcal{D}$, the class of isomorphisms in $\mathcal{D}$ satisfy the two-out-of-three condition. Thus if we look at the localization map $i : \mathcal{C} \to \mathcal{C}[W^{-1}]$, the morphisms in $\mathcal{C}$ which get sent to isomorphisms $\mathcal{C}[W^{-1}]$ will be a class $\widetilde{W}$ of morphisms containing W and all isomorphisms satisfying the two-out-of-three condition and the induced map $\mathcal{C}[\widetilde{W}^{-1}] \to \mathcal{C}[W^{-1}]$ coming from i will be an equivalence.

Let $(\mathcal{C}, W)$ be a category with weak equivalences. An explicit construction of $\mathcal{C}[W^{-1}]$ is given by Gabriel and Zisman [5] as follows: In general weak equivalences will not have inverses so two objects $X, Y \in \mathcal{C}$ which become isomorphic in $\mathcal{C}[W^{-1}]$ may only be related in $\mathcal{C}$ via a zig-zag

$$\begin{tikzcd}
X \arrow[r] & Z_1 & Z_2 \arrow[l] \arrow[r] & \cdots & Z_\ell \arrow[l] \arrow[r] & Y
\end{tikzcd}$$

of weak equivalences. Thus the objects in $\mathcal{C}[W^{-1}]$ are the same as the objects of $\mathcal{C}$, and morphisms in $\mathcal{C}[W^{-1}]$ are zig-zags of morphisms in $\mathcal{C}$ modulo equivalence where

- adjacent arrows pointing in the same direction may be composed
- adjacent arrows $\xleftarrow{w} \cdot \xrightarrow{w}$ or $\xrightarrow{w} \cdot \xleftarrow{w}$ labelled by a weak equivalence w may be removed

- all identity arrows pointing in either direction may be removed

The localization morphism $\mathcal{C} \to \mathcal{C}[W^{-1}]$ is then given by the identity on objects and by sending a morphism to the corresponding length one zig-zag.

This construction is difficult to work with in practice. Model categories introduce two auxiliary collections of morphisms, called fibrations and cofibrations, which help give an alternative description of $\mathcal{C}[W^{-1}]$.

3.2 Definition of a model category

We introduce some definition that are necessary for defining a model category.

Definition 3.2.1. Let $\mathcal{C}$ be a category and $p : A \to B$, $i : C \to D$ morphisms in $\mathcal{C}$. We say that i has the *left lifting property with respect to p*, or equivalently that p has the *right lifting property with respect to i*, if for every commutative diagram

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "i"'] & A \arrow[d, "p"] \\
D \arrow[r] & B
\end{tikzcd}$$

there exists a diagonal map

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "i"'] & A \arrow[d, "p"] \\
D \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

making the diagram commute.

Given two collections of maps S, T we say that S has the left lifting property with respect to T, or equivalently that T has the right lifting property with respect to S, if every $s \in S$ has the left lifting property with respect to every $t \in T$.

Given a collection of maps F, write $l(F)$ (write $r(F)$) for the collection of all maps having the left (right) lifting property with respect to every $f \in F$.

Definition 3.2.2. Let $\mathcal{C}$ be a category. An object U is a *retract* of an object X if there exists a commutative diagram

$$\begin{tikzcd}
U \arrow[r, "i"] \arrow[rr, bend left, "\mathrm{id}_U"] & X \arrow[r, "r"] & U.
\end{tikzcd}$$

We say that a morphism $f : U \to V$ is a retract of a morphism $g : X \to Y$ if it is a retract of g in the category of morphisms, i.e. if there exists a commutative diagram

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d] \arrow[dd, bend right, "\mathrm{id}_U"'] & V \arrow[d] \arrow[dd, bend left, "\mathrm{id}_V"] \\
X \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\
U \arrow[r, "f"] & V.
\end{tikzcd}$$

Finally, we say that a collection F of morphisms in $\mathcal{C}$ is *closed under retracts* if any morphism f which is a retract of some $g \in F$ belongs to F.

For model categories, we want to be able to pick a distinguished subclass of objects in $\mathcal{C}$ to represent the objects of the localized category $\mathcal{C}[W^{-1}]$. For this, we need a robust way to factor arbitrary morphisms through these such objects. The general tool we will use is the notion of a factorization system.

Definition 3.2.3. Let $\mathcal{C}$ be a category. A *factorization system* in a pair (A, B) of collections of morphisms in $\mathcal{C}$ satisfying the following properties:

(i) A and B are closed under retracts

(ii) $A \subseteq l(B)$ (equivalently $B \subseteq r(A)$)

(iii) every morphism f in $\mathcal{C}$ can be written factorially as a composition $f = pi$ where $i \in A$ and $p \in B$

Remark 3.2.4. It turns out that in conjunction with (iii), assumptions (i) and (ii) in Definition 3.2.3 are equivalent to $A = l(B)$ (equivalently $B = r(A)$).

Remark 3.2.5. The condition that the factorization be functorial in Definition 3.2.3 is potentially non-standard, however since we are only using this definition for model categories where we want functorial factorizations, we will make this assumption.

We are now ready to define a model category.

Definition 3.2.6. A *model category* is a locally small category with weak equivalences $(\mathcal{C}, W)$ along with two other classes of morphisms Fib and Cof satisfying the following axioms:

(i) $\mathcal{C}$ has all finite limits and colimits

(ii) (Cof, Fib $\cap$ W) and (Cof $\cap$ W, Fib) are weak factorization systems

The morphisms in Fib and Cof are referred to as *fibrations* and *cofibrations* respectively. The morphisms in Fib$\cap$ W and Cof$\cap$ W are referred to as *trivial fibrations* and *trivial cofibrations* respectively.

Example 3.2.7. The canonical example one should have in mind is $\mathcal{C} = \mathrm{Top}$ with the usual weak equivalences, Serre fibrations as fibrations, and cofibrations "generated" by the boundary inclusions $S^{n-1} \hookrightarrow D^n$.

Since $\mathcal{C}$ has all finite limits and colimits, $\mathcal{C}$ has an initial object $\emptyset$ and a final object $*$. We say that an object A is *fibrant* if the map $A \to *$ is a fibration and that A is *cofibrant* if the map $\emptyset \to A$ is a cofibration.

Proposition 3.2.8. *The composition of two fibrations is a fibration, similarly for cofibrations.*

Proof. We show this for fibrations. The proof for cofibrations is similar.

Let $f : X \to Y$ and $g : Y \to Z$ be two fibrations. By Remark 3.2.4, it suffices to show that $g \circ f$ has the right lifting property with respect to every trivial cofibration. Suppose we have a diagram of solid arrows

$$\begin{tikzcd}
A \arrow[r] \arrow[dd, "i"'] & X \arrow[d, "f"] \\
& Y \arrow[d, "g"] \\
B \arrow[r] \arrow[ru, dashed] \arrow[ruu, dashed] & Z,
\end{tikzcd}$$

where i is a trivial cofibration. Since g is a fibration, it the dashed arrow $B \to Y$ exists. Then since f is a fibration the second dashed arrow exists making the diagram commute. Thus we've shown $g \circ f$ has the right lifting property with respect to any trivial cofibration. ◻

Corollary 3.2.9. *If Y is fibrant and $X \to Y$ is a fibration, then X is fibrant. Similarly, if X is cofibrant and $X \to Y$ is a cofibration then Y is cofibrant.*

Proof. If Y is fibrant, then $Y \to *$ is a fibration. Thus the composite $X \to Y \to *$ is a fibration by Proposition 3.2.8 so X is fibrant. Similarly for cofibrant objects. ◻

Now, because we have functorial factorizations in a model category, we may functorially factor the maps $\emptyset \to X$ into a cofibration followed by a trivial fibration. That is, there exists a functor $Q : \mathcal{C} \to \mathcal{C}$ and a natural transformation $Q \Rightarrow \mathrm{id}$ such that

$$\emptyset \to X = \emptyset \to Q(X) \to X$$

where $Q(X) \to X$ is a trivial fibration and $\emptyset \to Q(X)$ is a cofibration. In particular, we have that $Q(X)$ is cofibrant and X is weakly equivalent to $Q(X)$. We call $Q(X)$ a *cofibrant replacement* of X. In a similar manner, there exists a *fibrant replacement* functor.

Proposition 3.2.10. *Let $Q : \mathcal{C} \to \mathcal{C}$ and $F : \mathcal{C} \to \mathcal{C}$ be cofibrant and fibrant replacement functors, respectively, as constructed above. Then $FQ : \mathcal{C} \to \mathcal{C}$ and $QF : \mathcal{C} \to \mathcal{C}$ are both* fibrant-cofibrant *replacement functors.*

Proof. We have a factorization

$$\emptyset \xrightarrow[\text{cofib}]{} Q(F(X)) \xrightarrow[\text{triv fib}]{} F(X).$$

Thus $Q(F(X))$ is weakly equivalent to $F(X)$, which is weakly equivalent to X. Moreover, $Q(F(X))$ is cofibrant, but it is also fibrant by Corollary 3.2.9 since $F(X)$ is fibrant and we have a fibration $Q(F(X)) \to F(X)$. The proof is similar for FQ. □

The benefit of model categories is that it will allow us to represent $\mathcal{C}[W^{-1}]$ as a category formed by identifying morphisms "up to homotopy" rather than having to present morphisms as zig-zags of maps. In general, weakly equivalent objects may only have an equivalence going one direction. However, between weakly equivalent fibrant-cofibrant objects, we can always produce a weak equivalence going both directions.

Proposition 3.2.11. *Let $f : X \to Y$ be a weak equivalence between two fibrant-cofibrant objects. Then there exists a weak equivalence $Y \to X$.*

Proof. Consider the factorization

$$\begin{tikzcd} & A \arrow[dr, "\text{fib}"] & \\ X \arrow[ur, "\text{triv cofib}", "\simeq"'] \arrow[rr, "\simeq", "f"'] & & Y. \end{tikzcd}$$

By the two-out-of-three property for weak equivalences, we have that the map $A \to Y$ is also a weak equivalence. Since Y is fibrant, so is A, and since X is cofibrant, so is A. Thus we have reduced to showing that trivial fibrations and trivial cofibrations between fibrant-cofibrants have weak equivalences going the other direction.

Suppose that $f : X \to Y$ is a trivial cofibration. Then the dashed arrow filling

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d] \\ A \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

gives such an equivalence. A similar proof works for trivial fibrations. □

We will upgrade this proposition in the sequel after we have defined an appropriate notion of homotopy between maps.

3.3 Homotopies and representations of the localization

In this section our goal is to use the model structure on a category $\mathcal{C}$ to abstract the notion of homotopy.

Definition 3.3.1. Let $X \in \mathcal{C}$. A *cylinder* for X is a factorization of the form

$$\begin{tikzcd} X \amalg X \arrow[r, "\partial_0 \amalg \partial_1"] \arrow[rr, bend left, "\mathrm{id} \amalg \mathrm{id}"] & IX \arrow[r, "\sigma"] & X \end{tikzcd}$$

where $\partial_0 \amalg \partial_1$ is a cofibration and σ is a weak equivalence.

A *cocylinder* for X is a factorization

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[rr, bend left, "(\mathrm{id},\mathrm{id})"] & X^I \arrow[r, "(d^0,d^1)"] & X \times X \end{tikzcd}$$

where s is a weak equivalence and (d^0, d^1) is a fibration.

Example 3.3.2. One should compare this with traditional topology. When $\mathcal{C} = \mathrm{Top}$ with the standard weak equivalences and Serre fibrations as fibrations, then $IX = [0,1] \times X$ and $X^I = \mathrm{Map}([0,1], X)$ are cyclinders and cocylinders of X where $X \amalg X \xrightarrow{\partial_0 \amalg \partial_1} [0,1] \times X$ is the inclusion of the copies $\{0\} \times X$ and $\{1\} \times X$ of X, and $\mathrm{Map}([0,1], X) \xrightarrow{(d^0,d^1)} X \times X$ is evaluation at 0 and 1.

Definition 3.3.3. Let $f_0, f_1 : A \to X$ be morphisms.

- A *left homotopy* from f_0 to f_1 is a morphims $h : IA \to X$ such that $h\partial_0 = f_0$ and $h\partial_1 = f_1$.
- A *right homotopy* from f_0 to f_1 is a morphism $h : A \to X^I$ such that $d^0 h = f_0$ and $d^1 h = f_1$.

A priori, the existence of a right homotopy is independent of the existence of a left homotopy. Additionally, it also appears that the choice of cylinder and cocylinder matters when deciding whether there exists a homotopy with that domain or codomain. In nice scenarios, both of these issues go away.

Lemma 3.3.4. *Let $f_0, f_1 : A \to X$ be maps with A cofibrant and X fibrant. The following are equivalent:*

(i) Then there exists a left homotopy from f_0 to f_1.

(ii) There exists a right homotopy from f_0 to f_1.

(iii) For any cylinder IA of A, there exists a left homotopy from f_0 to f_1 with domain IA.

(iv) For any cocylinder X^I of X, there exists a right homotopy from f_0 to f_1 with codomain X^I.

Proof. See [4, Lemma 2.2.12]. □

In the setting of Lemma 3.3.4, we will simply say that f_0 is homotopic to f_1 if f_0 is left (equiv. right) homotopic to f_1.

Lemma 3.3.5. *Let A be cofibrant and X be fibrant. Then the relation $\simeq$ on $\mathrm{Hom}_{\mathcal{C}}(A, X)$ given by $f_0 \simeq f_1$ if and only if f_0 is homotopic to f_1 is an equivalence relation.*

Proof. We have that $f \simeq f$ via $sf : A \to X^I$ where s is an in the definition of a cocylinder for X.

Now, suppose that $f_0 \simeq f_1$ via $h : IA \to X$ and the cylinder

$$\begin{tikzcd}
A \amalg A \arrow[r, "\partial_0 \amalg \partial_1"] \arrow[rr, bend left, "\mathrm{id} \amalg \mathrm{id}"] & IA \arrow[r, "\sigma"] & A.
\end{tikzcd} \tag{3.3.1}$$

Then

$$\begin{tikzcd}
A \amalg A \arrow[r, "\partial_1 \amalg \partial_0"] \arrow[rr, bend left, "\mathrm{id} \amalg \mathrm{id}"] & IA \arrow[r, "\sigma"] & A
\end{tikzcd}$$

is still a cylinder of A since $\partial_1 \amalg \partial_0$ is the cofibration $\partial_0 \amalg \partial_1$ composed with the isomorphism which swaps factors of $A \amalg A$, and all isomorphisms are trivial fibrations and trivial cofibrations. Now, with respect to this cylinder h gives a homotopy $f_1 \simeq f_0$.

Finally, suppose $f_0 \simeq f_1$ and $f_1 \simeq f_2$. Fix a cylinder as in (3.3.1) and let $h_0 : IA \to X$ and $h_1 : IA \to X$ witness $f_0 \simeq f_1$ and $f_1 \simeq f_2$ respectively. Now form a pushout diagram

$$\begin{tikzcd}
A \arrow[r, "\partial_1"] \arrow[d, "\partial_0"'] & IA \arrow[d, "i_0"] \\
IA \arrow[r, "i_1"] & I'A
\end{tikzcd}$$

which morally represents gluing two copies of IA to each other along $\{1\} \times A$ and $\{0\} \times A$. Then there exists a unique map $\sigma' : I'A \to A$ such that $\sigma' i_1 = \sigma' i_0 = \sigma$. We then claim that

$$\begin{tikzcd}
A \amalg A \arrow[r, "i_0\partial_0 \amalg i_1\partial_1"] \arrow[rr, bend left, "(\mathrm{id},\mathrm{id})"] & I'A \arrow[r, "\sigma'"] & A
\end{tikzcd}$$

is a cylinder. If so we are done as the map $h' : I'A \to X$ defined by $h' i_0 = h_0$ and $h' i_1 = h_1$ defines a homotopy $f_0 \simeq f_2$. For this, notice that ∂_0 and ∂_1 are trivial cofibrations. Thus i_0 and i_1 are trivial cofibrations so σ' is a weak equivalence by the two-out-of-three property. To show that $i_0\partial_0 \amalg i_1\partial_1$ is a cofibration we have a diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\partial_0"'] & A \amalg A \arrow[r, "\partial_0 \amalg \partial_1"] \arrow[d, "\mathrm{id} \amalg \partial_0"'] & IA \arrow[d, "i_0"] \\
IA \arrow[r] & A \amalg IA \arrow[r, "i_0\partial_0 \amalg i_1"] & I'A
\end{tikzcd}$$

where the outer and left-most squares are coCartesian. Thus the right-most square is coCartesian so $i_0\partial_0 \amalg i_1$ is a cofibration as $\partial_0 \amalg \partial_1$ is. Thus composing with the cofibration $\mathrm{id} \amalg \partial_1$ we see that $i_0\partial_0 \amalg i_1\partial_1$ is a cofibration. □

Definition 3.3.6. Given A cofibrant and X fibrant, write $[A, X] = \mathrm{Hom}_{\mathcal{C}}(A, X)/\simeq$ for the morphisms $A \to X$ up to homotopy.

Let $\mathcal{C}_c$ denote the full subcategory of $\mathcal{C}$ spanned by cofibrant objects and let $\mathcal{C}_f$ denote the full subcategory of $\mathcal{C}$ spanned by fibrant objects. Right homotopies are compatible with composition on the right and left homotopies are compatible with composition on the left, and thus we get a functor

$$[-,-] : \mathcal{C}_c^{\mathrm{op}} \times \mathcal{C}_f \to \mathrm{Set}.$$

We are now ready to present the two primary theorems on how model categories give us a handle on $\mathcal{C}[W^{-1}]$.

Definition 3.3.7. A morphisms $f : X \to Y$ is a *homotopy equivalence* if there exists a morphism $g : Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$.

Theorem 3.3.8 (Categorical Whitehead). *Let $f : X \to Y$ be a weak equivalence between two fibrant-cofibrant objects. Then f is a homotopy equivalence.*

Proof. Following the steps of Proposition 3.2.11, we see that it suffices to prove the theorem for trivial fibrations and trivial cofibrations. We do the case of trivial cofibrations as the other case is dual.

Again following the proof of Proposition 3.2.11, we in fact have a genuine left inverse $g : Y \to X$ of f. Thus it suffices to show that $fg \simeq \mathrm{id}_Y$. Consider then a cocylinder

$$Y \xrightarrow{s} Y^I \xrightarrow{(d^0,d^1)} Y \times Y.$$

Then we have a filler

$$\begin{tikzcd}
X \arrow[r, "sf"] \arrow[d, "\text{triv cof}\ f"'] & Y^I \arrow[d, "\text{fib}"', "{(d^0,d^1)}"] \\
Y \arrow[r, "{(\mathrm{id},fg)}"'] \arrow[ur, dashed, "h"] & Y \times Y
\end{tikzcd}$$

and h is a homotopy $fg \simeq \mathrm{id}_Y$ as required. $\square$

Example 3.3.9. When $\mathcal{C} = \mathrm{Top}$ in the standard model structure, i.e. with weak equivalences and Serre fibrations, CW complexes are fibrant-cofibrant. Thus in this example Whitehead's theorem becomes a special case of Theorem 3.3.8.

Theorem 3.3.10. *Let $\mathcal{C}_{cf}$ denote the full subcategory of $\mathcal{C}$ spanned by fibrant-cofibrant objects. Denote by $\pi(\mathcal{C}_{cf})$ the category whose objects are the same as $\mathcal{C}_{cf}$ and whose morphisms are given by* $\mathrm{Hom}_{\pi(\mathcal{C}_{cf})}(X, Y) = [X, Y]$. *Then the following hold:*

(i) Let $F : \mathcal{C} \to \mathcal{C}_{cf}$ denote a fibrant-cofibrant replacement functor. Then the composite $\mathcal{C} \xrightarrow{F} \mathcal{C}_{cf} \to \pi(\mathcal{C}_{cf})$ is a localization $\mathcal{C} \to \mathcal{C}[W^{-1}]$.

(ii) The inclusion $\mathcal{C}_{cf} \subseteq \mathcal{C}$ induces an equivalence $\pi(\mathcal{C}_{cf}) \simeq \mathcal{C}[W^{-1}]$.

Proof (Sketch). For (i), let $i : \mathcal{C} \xrightarrow{F} \mathcal{C}_{cf} \to \pi(\mathcal{C}_{cf})$. By Theorem 3.3.8, we have that i sends weak equivalences to isomorphisms. Thus it just remains to check the universal property for i.

Let $G : \mathcal{C} \to \mathcal{D}$ be a functor which sends weak equivalences to isomorphisms. We have a natural transformation $\mathrm{id} \Rightarrow F$ such that each $X \to F(X)$ is a weak equivalence. Composing with G, we find that $G \cong GF = G|_{\mathcal{C}_{cf}} F$. Thus we get that G factors through F.

Next, suppose $f, g : X \to Y$ with $f \simeq g$ via a homotopy $h : IX \to Y$. Letting ∂_0, ∂_1 and σ be as in Defintion 3.3.1, then $\sigma\partial_0 = \mathrm{id} = \sigma\partial_1$. Since σ is a weak equivalence, $G\sigma$ is an isomorphism so

$$G\sigma \circ G\partial_0 = G\sigma \circ G\partial_1 \Longrightarrow G\partial_0 = G\partial_1.$$

Thus

$$Gf = Gh \circ G\partial_0 = Gh \circ G\partial_1 = Gg$$

so $G|_{\mathcal{C}_{cf}}$ factors through $\mathcal{C}_{cf} \to \pi(\mathcal{C}_{cf})$. In total, G factors through the composite $i : \mathcal{C} \xrightarrow{F} \mathcal{C}_{cf} \to \pi(\mathcal{C}_{cf})$. It is then easy to check that this construction shows that pre-composition with i gives an equivalence between functors $\pi(\mathcal{C}_{cf}) \to \mathcal{D}$ and functors $\mathcal{C} \to \mathcal{D}$ sending weak equivalences to isomorphisms.

(ii) follows from (i). $\square$

Remark 3.3.11. (ii) in Theorem 3.3.10 holds even without functorial factorization systems. For details on this, see [4].

3.4 Derived functors

3.4.1 Kan extensions

In this section we introduce a foundational concept in category theory which will be of special interest for us when studying localized categories.

Suppose we have a functor $F : \mathcal{C} \to \mathcal{D}$ and another functor $K : \mathcal{C} \to \mathcal{C}'$. What is the universal way to "extend" F to $\mathcal{C}'$ along K? If K is an inclusion, this is motivated by wanting an actual extension of F to $\mathcal{C}'$, though for more general K we are asking for a universal way factor F through K. There are two ways to formalize this question which we present now.

Definition 3.4.1. Let $F : \mathcal{C} \to \mathcal{D}$ and $K : \mathcal{C} \to \mathcal{C}'$ be functors. A *left Kan extension* of F along K is a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \Downarrow \eta & \mathcal{D} \\
& \mathcal{C}' \arrow[ur, "\mathrm{Lan}_K(F)"'] &
\end{tikzcd}$$

commuting only up to the natural transformation η that is universal among all such diagrams, i.e. given another we can write

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \Downarrow \varepsilon & \mathcal{D} & = & \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \eta \Downarrow & \mathcal{D} \\
& \mathcal{C}' \arrow[ur, "G"'] & & & & \mathcal{C}' \arrow[ur, bend left, "\mathrm{Lan}_K(F)", ""{name=L}] \arrow[ur, bend right=50, "G"', ""{name=G}] \arrow[from=L, to=G, Rightarrow, "\exists!"] &
\end{tikzcd}.$$

A *right Kan extension* of F along K is a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \Uparrow \eta & \mathcal{D} \\
& \mathcal{C}' \arrow[ur, "\mathrm{Ran}_K(F)"'] &
\end{tikzcd}$$

commuting up to ε that is universal among all such diagrams, i.e. given another we can write

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Uparrow\varepsilon"] & \mathcal{D} \\
& \mathcal{C}' \arrow[ur, "G"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\eta\Uparrow"] & \mathcal{D} \\
& \mathcal{C}' \arrow[ur, bend left, "\mathrm{Ran}_K(F)"] \arrow[ur, bend right=50, "G"'] \arrow[ur, phantom, bend right=20, "\overset{\exists!}{\Leftarrow}"] &
\end{tikzcd}.$$

Despite being defined by a universal property, Kan extensions are somewhat misbehaved. In fact, even if $K : \mathcal{C} \hookrightarrow \mathcal{C}'$ is an inclusion, then a Kan extension if it exists may not be an extension and vice versa, i.e. we do not necessarily have that $\mathrm{Lan}_K(F) \circ K = F$ or $\mathrm{Ran}_K(F) \circ K = F$.

To get a better behaved theory, it is nice to enforce that our Kan extensions be preserved under certain functors.

Definition 3.4.2. Let $G : \mathcal{D} \to \mathcal{E}$ be a functor. We say that a left Kan extension

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{D} \\
& \mathcal{C}' \arrow[ur, "\mathrm{Lan}_K(F)"'] &
\end{tikzcd} \tag{3.4.1}$$

is *preserved* under G if the composite diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{D} \arrow[r, "G"] & \mathcal{E} \\
& \mathcal{C}' \arrow[ur, "\mathrm{Lan}_K(F)"'] & &
\end{tikzcd}
\cong
\begin{tikzcd}
\mathcal{C} \arrow[rr, "GF"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow G\eta"] & \mathcal{E} \\
& \mathcal{C}' \arrow[ur, "G\mathrm{Lan}_K(F)"'] &
\end{tikzcd}$$

is still a left Kan extension. Similarly for right Kan extensions.

Given a functor $G : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}$ (i.e. a contravariant functor $\mathcal{D} \to \mathcal{E}$), we say that a left Kan extension as in (3.4.1) is preserved by G if the right Kan extension

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[rr, "F^{\mathrm{op}}"] \arrow[dr, "K^{\mathrm{op}}"'] & \arrow[d, phantom, "\Uparrow\eta^{\mathrm{op}}"] & \mathcal{D}^{\mathrm{op}} \\
& (\mathcal{C}')^{\mathrm{op}} \arrow[ur, "\mathrm{Lan}_K(F)^{\mathrm{op}}"'] &
\end{tikzcd}$$

induced by (3.4.1) is preserved by G. Similarly for right Kan extensions.

Definition 3.4.3. We say that a right Kan extension of $F : \mathcal{C} \to \mathcal{D}$ is *pointwise* if it is preserved by all representable functors $\mathrm{Hom}_{\mathcal{D}}(d, -)$. We say that a left Kan extension of $F : \mathcal{C} \to \mathcal{D}$ is pointwise if it preserved by all corepresentable functors $\mathrm{Hom}_{\mathcal{D}}(-, d)$.

We say that a Kan extension is *absolute* if it is preserved by all functors.

In the pointwise setting, Kan extensions are well behaved. Indeed, at least up to isomorphism, pointwise Kan extensions along fully faithful inclusions are indeed extensions and pointwise Kan extensions are commuted via limits and colimits. For details on this see [10].

3.4.2 Derived functors

We now return to a discussion of model categories. Let $(\mathcal{C}, W)$ be a category with weak equivalences. We will sometimes write $\mathrm{Ho}(\mathcal{C})$ in place of $\mathcal{C}[W^{-1}]$ and refer to $\mathrm{Ho}(\mathcal{C})$ as the *homotopy category* of $\mathcal{C}$.

Suppose we have an arbitrary functor $F : \mathcal{C} \to \mathcal{D}$ between two categories with weak equivalences. We may ask: What is the universal homotopy invariant functor associated to F? One way to formalize this is by asking for the left (right) Kan extension of $\mathcal{C} \xrightarrow{F} \mathcal{D} \to \mathrm{Ho}(\mathcal{D})$ along $\mathcal{C} \to \mathrm{Ho}(\mathcal{C})$. Pictorially, we are considering the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[dr] & \mathcal{D} \arrow[r] \arrow[d, phantom, "\Downarrow"] & \mathrm{Ho}(\mathcal{D}) \\
& \mathrm{Ho}(\mathcal{C}) \arrow[ur, "\mathbf{L}F"'] &
\end{tikzcd}$$

Definition 3.4.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between two categories with weak equivalences. The *left derived functor* of F, denoted $\mathbf{L}F$, is the right Kan extension of $\mathcal{C} \xrightarrow{F} \mathcal{D} \to \mathrm{Ho}(\mathcal{D})$ along $\mathcal{C} \to \mathrm{Ho}(\mathcal{D})$.

The *right derived functor* of F, denoted $\mathbf{R}F$, is the left Kan extension of $\mathcal{C} \xrightarrow{F} \mathcal{D} \to \mathrm{Ho}(\mathcal{D})$ along $\mathcal{C} \to \mathrm{Ho}(\mathcal{D})$.

It turns out that by enforcing factorial factorizations in our model categories, left and right derived functors have particularly simple forms in many common scenarios. In fact, in these scenarios, they even have lifts to functors $\mathcal{C} \to \mathcal{D}$ (note that every functor $\mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ has a lift to $\mathcal{C} \to \mathrm{Ho}(\mathcal{D})$ but it should be surprising that there exists a lift along $\mathcal{D} \to \mathrm{Ho}(\mathcal{D})$).

Theorem 3.4.5. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories with weak equivalences and let $\delta : \mathcal{D} \to \mathrm{Ho}(\mathcal{D})$ be localization.*

(i) Suppose that F sends trivial cofibrations between cofibrant objects to weak equivalences. Then if $Q : \mathcal{C} \to \mathcal{C}$ is a cofibrant replacement functor, the functor $\delta FQ : \mathcal{C} \to \mathrm{Ho}(\mathcal{D})$ descends to a morphism

$\partial FQ : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ which is an absolute right Kan extension of ∂F along $\mathcal{C} \to \mathrm{Ho}(\mathcal{C})$. In particular, $\mathbf{L}F = \partial FQ$*.*

(ii) Suppose that F sends trivial fibrations between fibrant objects to weak equivalences. Then if $R : \mathcal{C} \to \mathcal{C}$ is a fibrant replacement functor, the functor $\partial FR : \mathcal{C} \to \mathrm{Ho}(\mathcal{D})$ descends to a morphism $\partial FR : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ which is an absolute left Kan extension of ∂F along $\mathcal{C} \to \mathrm{Ho}(\mathcal{C})$. In particular, $\mathbf{R}F = \partial FR$*.*

Proof. We prove (i) as (ii) is dual. Let Q be a cofibrant replacement functor. Then since factorizations are functorial we get a natural transformation $q : Q \Rightarrow \mathrm{id}$ such that $q_X : QX \to X$ is a weak equivalence for all X. Thus given a weak equivalence $X \xrightarrow{\simeq} Y$, the diagram

$$\begin{tikzcd}
QX \arrow[r] \arrow[d, "\simeq"] & QY \arrow[d, "\simeq"] \\
X \arrow[r, "\simeq"] & Y
\end{tikzcd}$$

shows that $QX \to QY$ is a weak equivalence by the two-out-of-three property.

Now, a consequence of Ken Brown's lemma (Theorem 3.4.6) which we state below is that F preserves *all* weak equivalences between cofibrant objects not just trivial cofibrations. Thus we see that FQ sends weak equivalences to weak equivalences, and hence ∂FQ descends to a map $\partial FQ : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$. We then claim that $(\partial FQ, \partial Fq : \partial FQ \Rightarrow \partial F)$ is a right Kan extension. For this, suppose that we have a functor $\mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ which is induced by a functor $L : \mathcal{C} \to \mathrm{Ho}(\mathcal{D})$ with a natural transformation $\ell : L \Rightarrow \partial F$. Because L sends weak equivalences to isomorphisms we have that $Lq : LQ \Rightarrow L$ is a natural isomorphism. Thus we have a factoring

$$\begin{tikzcd}
L \arrow[r, Rightarrow, "(Lq)^{-1}"] \arrow[rrr, Rightarrow, bend right, "\ell"'] & LQ \arrow[r, Rightarrow, "\ell Q"] & \partial FQ \arrow[r, Rightarrow, "\partial Fq"] & \partial F
\end{tikzcd}$$

of ℓ through ∂Fq which one checks is unique.

To see that this extension is absolute, suppose we have another functor $H : \mathrm{Ho}(\mathcal{D}) \to \mathcal{E}$. Then applying (i) to the functor $\mathcal{C} \xrightarrow{\partial F} \mathrm{Ho}(\mathcal{D}) \xrightarrow{H} \mathcal{E}$ where the weak equivalences of $\mathcal{E}$ are isomorphisms, we find that $(H\partial FQ, H\partial Fq : H\partial FQ \Rightarrow H\partial F)$ is a right Kan extension, as required. $\square$

We now state the theorem that was mentioned in the above proof.

Theorem 3.4.6 (Ken Brown's Lemma)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a morphisms between two categories with weak equivalences. If F sends trivial cofibrations (trivial fibrations) between cofibrant (fibrant)* *objects to weak equivalences, then F sends all weak equivalences between cofibrant (fibrant) objects to weak equivalences.*

Proof. See [4, Proposition 2.2.7]. ☐

Thus we see that in the setting of Theorem 3.4.5 derived functors have lifts, i.e. we can always build a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widetilde{F}"] \arrow[d] & \mathcal{D} \arrow[d] \\
\mathrm{Ho}(\mathcal{C}) \arrow[r, "\mathbf{L}F\ (\text{or } \mathbf{R}F)"'] & \mathrm{Ho}(\mathcal{D}).
\end{tikzcd}$$

We also have a corollary coming from the construction of these derived functors.

Corollary 3.4.7. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ send trivial cofibrations (trivial fibrations) between cofibrant (fibrant) objects to weak equivalences. Moreover, suppose that F preserves cofibrant (fibrant) objects. Then the canonical comparison $\mathbf{L}G \circ \mathbf{L}F \Rightarrow \mathbf{L}(G \circ F)$ (resp. $\mathbf{R}(G \circ F) \Rightarrow \mathbf{R}G \circ \mathbf{R}F$) is a natural isomorphism.*

3.5 Relation to homological algebra

We now explain the relation of the general theory of model categories to the some standard procedures in homological algebra. In homological algebra, one often deals with chain complexes of R-modules and computes derived functors by taking injective or projective resolutions. Let $\mathcal{A} = R-\mathrm{Mod}$ denote the category of R-modules for some commutative ring R. There exists two standard model structures relevant to homological algebra which we state below.

Theorem 3.5.1. *There exists a model structure on $\mathrm{Ch}_{\geq 0}(\mathcal{A})$, the non-negatively supported chain complexes of R-modules, such that*

- *the weak equivalences are quasi-isomorphisms*
- *the fibrations are maps which are epimorphisms in each positive degree*
- *the cofibrations are maps which are monomorphisms with projective cokernel in each degree*

called the projective model structure.

Theorem 3.5.2. *There exists a model structure on $\mathrm{Ch}^{\geq 0}(\mathcal{A})$, the non-negatively supported cochain complexes of R-modules, such that*

- *the weak equivalences are quasi-isomorphisms*
- *the cofibrations are maps which are monomorphisms in each positive degree*
- *the fibrations are maps which are epimorphisms with injective kernel in each degree*

called the injective model structure.

It is clear that in the injective model structure the fibrant objects are those complexes whose terms are injective modules, and the cofibrant objects in the projective model structure are those complexes whose terms are projective models.

Proposition 3.5.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between module categories. Then F induces a map $\mathrm{Ch}^{\geq 0}(F) : \mathrm{Ch}^{\geq 0}(\mathcal{A}) \to \mathrm{Ch}^{\geq 0}(\mathcal{B})$ which sends weak equivalences between fibrant objects to weak equivalences in the injective model structure. Similarly, $\mathrm{Ch}_{\geq 0}(F) : \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{B})$ sends weak equivalences between cofibrant objects to weak equivalences in the projective model structure.*

Proof. If $I^\bullet$ and $J^\bullet$ are fibrant, i.e. complexes of injective objects, then any quasi-isomorphism $f : I^\bullet \to J^\bullet$ is in fact a homotopy equivalence and thus preserved by F. Similarly for the projective model structure. ◻

It follows that taking injective resolutions and projective resolutions then applying F is simply computing the right and left derived functors of the extension of F to (co)chain complexes in the respective model category by Theorem 3.4.5. The usual reason for assuming left exactness before taking right derived functors is to ensure that

$$H^0(\mathbf{R}F) = F$$

and similarly the usual assumption of right exactness before taking left derived functors is to ensure that

$$H_0(\mathbf{L}F) = F$$

even though the above shows that left/right exactness is not necessary for the existence of derived functors.

4 Basics of ∞-categories

In category theory, we have objects and morphisms. However, in many examples there exists a natural notion of "morphisms between morphisms." The most basic instance of this is the category of all categories Cat. Here the objects are (small) categories, the 1-morphisms are functors, and the 2-morphisms are natural transformations between functors, and the hierarchy stops here. In theory, however, we could have 3-morphisms between these 2-morphisms, 4-morphisms between these 3-morphisms and so on. Making what we mean by this rigorous is somewhat of a difficult task. One potential route is to define a notion of an n-category inductively.

Definition 4.0.1. A (strict) n-category is defined inductively as follows:

- A 1-category is a regular category.
- An n-category is a category enriched over $(n-1)$-categories, i.e. a category $\mathcal{C}$ such that for any two objects $A, B \in \mathcal{C}$, the collection of maps between them $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an $(n-1)$-category.

Example 4.0.2. The category Cat of all (1-)categories is a 2-category. Indeed, given two categories $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$, the collection of maps between them is naturally the 1-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of functors $\mathcal{C} \longrightarrow \mathcal{D}$ with natural transformations between them.

Example 4.0.3. We could turn the category Top of topological spaces into a 2-category where the 1-category $\mathrm{Hom}(X, Y)$ has objects continuous maps $X \longrightarrow Y$ and morphisms as homotopies between such maps. Allowing homotopies between homotopies, and so on, one could turn Top into an n-category for any n.

A fundamental issue with Definition 4.0.1 is that n must be finite—it does not give an answer as to how to define a category with k-morphisms for any $k \in \mathbb{N}$ nor is there an obvious fix to this. The theory of ∞-categories gives a rigorous way to let $n \longrightarrow \infty$ in Definition 4.0.1 as well as the tools to organize all the extra structure that comes with such an object.

4.1 Simplicial models

It turns out that the best way to represent ∞-categories is as simplicial sets. This method was first proposed and explored by Joyal and then later popularized and further developed by Lurie. In this section we discuss this approach.

4.1.1 Simplicial sets and the nerve of a category

Definition 4.1.1. The *simplex category* Δ is the category whose objects are *non-empty* finite totally ordered sets and whose morphisms are (non-strict) order preserving maps.

It turns out that up to equivalence we can write down a very explicit description of Δ.

Proposition 4.1.2. *Let Δ' be the category described as follows.*

- *The objects of Δ' are the finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for $n \in \mathbb{N}$.*
- *The morphisms of Δ' are generated by, i.e. finite compositions of, the following morphisms*
    - *the face maps δ_i^n where $\delta_i^n : [n-1] \longrightarrow [n]$ be the unique order preserving morphism whose image excludes i, $i = 0, \ldots, n$*
    - *the degeneracy maps σ_i^n where $\sigma_i^n : [n] \longrightarrow [n-1]$ is the surjection with $\sigma_i^n(i) = \sigma_i^n(i+1) = i$, $i = 0, \ldots, n-1$*

subject to the following relations

$$\begin{aligned}
\delta_i^{n+1} \circ \delta_j^n &= \delta_{j+1}^{n+1} \circ \delta_i^n && i \leq j \\
\sigma_j^n \circ \sigma_i^{n+1} &= \sigma_i^n \circ \sigma_{j+1}^{n+1} && i \leq j \\
\sigma_j^n \circ \delta_i^{n+1} &= \begin{cases} \delta_i^n \circ \sigma_{j-1}^{n-1} & i < j \\ \mathrm{id} & i = j \text{ or } i = j+1 \\ \delta_{i-1}^n \circ \sigma_j^{n-1} & j+1 < i. \end{cases}
\end{aligned}$$

Then the inclusion $\Delta' \hookrightarrow \Delta$ *is an equivalence.*

As a consequence of this proposition, when defining functors out of Δ or Δ^{op}, we only need to define the image of $[n]$ for each $n \in \mathbb{N}$ as well as the face and degeneracy maps (subject to certain relations).

Definition 4.1.3. A *simplicial set* is a functor $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$.

In light of Proposition 4.1.2, we can represent a simplicial set by a diagram

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X_2 \arrow[l, shift right=2] \arrow[l] \arrow[l, shift left=2] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_1 \arrow[l, shift right=1] \arrow[l, shift left=1] \arrow[r, shift left=1] \arrow[r, shift right=1] & X_0 \arrow[l]
\end{tikzcd}$$

where X_n denotes $X([n])$ and are referred to as the *n-simplices* of X and the right facing arrows represent the face maps, i.e. the duals of δ_i^n's, and the left facing arrows represent the degeneracy maps, i.e. the duals of σ_i^n's. Often, however, we will leave the degeneracy maps implicit and write simplicial sets as a diagram

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_1 \arrow[r, shift left=1] \arrow[r, shift right=1] & X_0.
\end{tikzcd}$$

With simplicial sets now defined, we move on to how to view a category as a simplicial set. Let $\mathcal{C}$ be a (small) category and denote by $\mathcal{C}_0$ and $\mathcal{C}_1$ the set of all objects of $\mathcal{C}$ and the set of all morphisms of $\mathcal{C}$, respectively. We have two natural maps

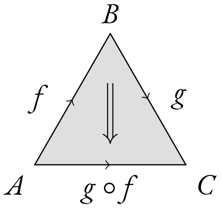
$$i, f : \mathcal{C}_1 \rightrightarrows \mathcal{C}_0$$

where $i(A \to B) = A$ picks out the domain and $f(A \to B) = B$ picks out the codomain. We also have a map $s : \mathcal{C}_0 \to \mathcal{C}_1$ given by $s(A) = \mathrm{id}_A$. The simplicial set conditions on face and degeneracy maps between zero and one simplices exactly demand that $is = \mathrm{id}$ and $fs = \mathrm{id}$, which in this case is true since $s(A) = \mathrm{id}_A : A \to A$.

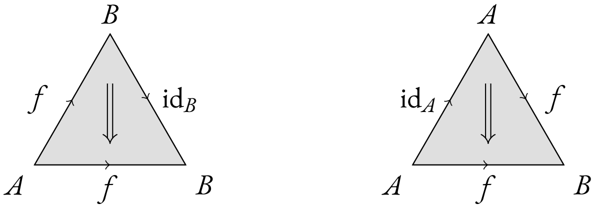
Thus given a category $\mathcal{C}$, we have built the zero and one simplices that form the start of a simplicial set, and the conditions on face and degenercy maps assert the existence of an endomorphism for each object (which will later be forced to satisfy the identity morphisms axioms). However, we still have not managed to capture the composition in $\mathcal{C}$. For this, we let $\mathcal{C}_2$ be the set of all length two chains of composable morphisms, i.e.

$$\mathcal{C}_2 = \{A \xrightarrow{f} B \xrightarrow{g} C : f, g \in \mathcal{C}_1\}.$$

This comes with three maps $\mathcal{C}_1$: one that picks out f, one that picks out g and finally one that picks out the composition $g \circ f$. Visually, $\mathcal{C}_2$ represents two simplices in a simplicial set we are building and we should think of the three edges as f, g and $g \circ f$, i.e.



Philosophically, the existence of this 2-simplex *witnesses* that the 1-simplex $g \circ f \in \mathcal{C}_1$ is a composition of the two 1-simplices $f, g \in \mathcal{C}_1$. We can also construct two degenerate 2-simplices out of an edge $f : A \to B$ as follows:



(4.1.1)

which gives us two degeneracy maps $\mathcal{C}_1 \rightrightarrows \mathcal{C}_2$. From this discussion, we are able to extend our simplicial diagram to include 2-simplices, i.e.

$$\begin{tikzcd} \mathcal{C}_2 \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & \mathcal{C}_1 \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathcal{C}_0 \arrow[l] \end{tikzcd} \tag{4.1.2}$$

and the fact that the face and degeneracy maps for 2-simplices satisfy the required relations is exactly saying that our identities given by the degeneracy map $\mathcal{C}_0 \to \mathcal{C}_1$ satisfy the identity axioms, i.e. $f \circ \mathrm{id} = f$ and $\mathrm{id} \circ f = f$. This can also been seen visually via the existence of the two degenerate 2-simplices in (4.1.1).

Now, following the philosophy above, compositions are *witnessed* by the existence of certain 2-simplices. That is, every 2-simplex witnesses some composition. Thus, our ability to compose any two compatible morphisms is not encoded simply in the structure of face and degeneracy maps in simplicial sets, but rather in an statement expressing that we have "sufficiently many" 2-simplices. In fact, in this specific scenario we have that

$$\mathcal{C}_2 = \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \tag{4.1.3}$$

where the two maps to $\mathcal{C}_0$ defining this fibre product are i, f. It is (4.1.3) which tells us that compositions exist as (4.1.3) says that for any two morphisms with compatible codomain and domain, there exists a *unique* 2-simplex witnessing their composition.

From (4.1.3) and the above discussion, we see that (4.1.2) in fact encodes all the information needed to fully recover our category $\mathcal{C}$. The fact that we can recover $\mathcal{C}$ using a truncated simplicial diagram is a shadow of the fact that there exists only objects and morphisms, and no higher morphisms. However, to put it on equal footing with the theory we will come to develop, it is beneficial to extend (4.1.2) to a full simplicial set. To do this we make the following definition:

Definition 4.1.4. Let $\mathcal{C}$ be a (small) category. Let $\mathcal{C}_0$ denote the set of objects of $\mathcal{C}$ and for all $n \geq 1$ let $\mathcal{C}_n$ denote the set of length n chains of composable morphisms, i.e.

$$\mathcal{C}_n = \{A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots A_{n-1} \xrightarrow{f_n} A_n : f_1, \ldots, f_n \in \mathcal{C}_1\}.$$

The *nerve* of $\mathcal{C}$, denoted $N(\mathcal{C})$, is the simplicial set

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \mathcal{C}_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{C}_1 \arrow[r, shift left=1] \arrow[r, shift right=1] & \mathcal{C}_0 \end{tikzcd}$$

where the face maps are given composing adjacent morphisms in a chain, and the degeneracy maps are given by inserting an identity morphism at different positions into the chain.

A tedious exercise is combinatorics shows that $N(\mathcal{C})$ is indeed a simplicial set, and it turns out that we can identify exactly which simplicial sets arise as the nerve of a category.

Theorem 4.1.5. *Let X be a simplicial set. Then X is isomorphic to the nerve of a category if and only if it satisfies the* Segal condition*: for all $n \geq 1$*

$$X_n \cong \underbrace{X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ times}}$$

via the natural induced maps coming from the face maps $X_n \longrightarrow X_1$ picking out the n different edges of an n-simplex.

Proof. From the definition of $N(\mathcal{C})$, if $X \cong N(\mathcal{C})$ for some $\mathcal{C}$ then it is clear that the segal condition holds.

Conversely, suppose that X satisfies the Segal condition. Then we may define a category $\mathcal{C}$ whose objects are given by X_0 and whose morphisms are given by X_1 where the two face maps $i, f : X_1 \rightrightarrows X_0$ pick out the domain and codomain of a morphism respectively. The Segal condition $X_2 \cong X_1 \times_{X_0} X_1$ tells us that for any two morphisms $f : x \to y$ and $g : y \to z$, there exists a unique 2-simplex corresponding to this pair. We define the composition $g \circ f$ in $\mathcal{C}$ to be the third edge of this unique 2-simplex. Moreover, we may define the identity morphism of an object $x \in X_0 = \mathcal{C}_0$ as the image of the degeneracy map $X_0 \to X_1$. The simplicial set conditions then imply that $\mathcal{C}$ is indeed a category.

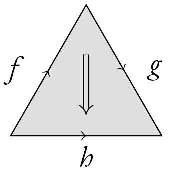
Finally, by definition of the nerve and construction of $\mathcal{C}$, we get an induced map $X \to N(\mathcal{C})$. The Segal condition on X_n for $n > 2$ then tells us that this induced map is indeed an isomorphism. □

Thus we have succeeded in viewing categories as special kinds of simplicial sets X, where via the Segal condition all higher simplices X_n are determined by the zero and one simplices. To define ∞-categories we will broaden the scope of which simplicial sets we consider and make use of higher simplices to represent higher morphisms.

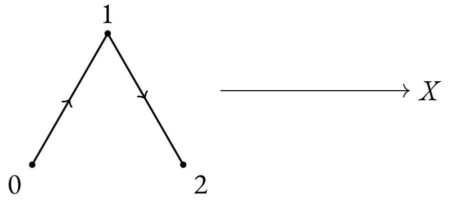
4.1.2 ∞-categories as simplicial sets

Let $X : \Delta^{\text{op}} \to \text{Set}$ be a simplicial set. We want to generalize our intuition from the previous section of constructing nerves of categories to decide a reasonable restriction on X to where it makes sense to refer to elements of X_n as *n-morphisms*. From §4.1.1, the existence of identities as well as composition (if it exists) behaving as expected is already encoded in the face and degeneracy relations defining a simplicial set. Thus the only concern is the *existence* of compositions which is statement about X having sufficiently many n-simplices for all $n \geq 2$.

Let us first consider the case of existence of compositions of 1-morphisms. From §4.1.1, a 2-simplex

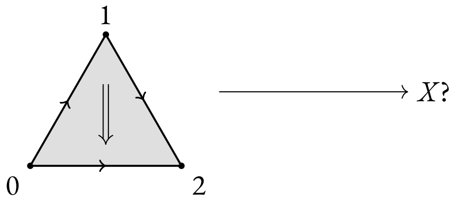


witnesses that h is a composition of f and g. Thus the question of whether we can compose any two compatible 1-morphisms boils down to the following question: Does every map

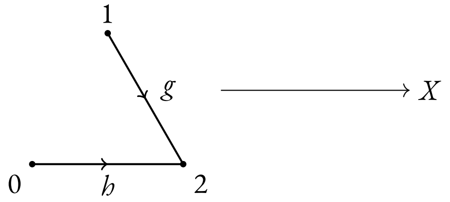


(4.1.4)

of simplicial sets extend to a map



Note that it is important that we are removing the edge opposite the vertex 1 in (4.1.4), as asking for a filler to a map



would be asking for a morphism f such that $g \circ f = h$, which of course need not always exists. Indeed, if $h = \mathrm{id}$ then this is asking for a right inverse to g. Similarly if we were to ask for fillers to maps of the triangle with the edge opposite the vertex 2 removed.

To encapsulate the domains we wish to have fillers of we make the following definition.

Definition 4.1.6. The *k-th horn* of the n-simplex for $0 \leq k \leq n$, denoted Λ^n_k, is the simplicial set given by removing the face opposite the k-th vertex in $\partial\Delta^n$.

The condition that 1-morphisms have compositions in X in then expressed by that statement that every map $\Lambda^2_i \to X$ for $0 < i < 2$ has an extension to a map $\Delta^2 \to X$, i.e. a dashed morphism in

$$\begin{tikzcd}
\Lambda^2_i \arrow[r, hook] \arrow[d] & \Delta^2 \arrow[dl, dashed] \\
X &
\end{tikzcd}$$

exists making the diagram commute. Note, however, the existence of such an extension is not assumed to be unique. That is, we are only asking that compositions exist, not that they are unique. In the special case of $N(\mathcal{C})$ we saw that they both exist and are unique thanks to the Segal condition, but this is a condition that it turns out is better to relax.

Generalizing to higher dimensional simplices, the correct formulation that n-morphisms have compositions ends up being that all *inner horns*, i.e. Λ^n_i for $0 < i < n$, extend to n-simplices.

Definition 4.1.7. An ∞-*category* is a simplicial set X such that every map $\Lambda^n_i \to X$ for $0 < i < n$ extends to a map $\Delta^n \to X$.

If we enforce that outer horns should also have fillings then we get an ∞-categorical notion of a groupoid, since we saw that fillings of outer horns correspond to the existence of left and right inverses.

Definition 4.1.8. A *Kan complex* or ∞-*groupoid* is a simplicial set X such that every map $\Lambda^n_i \to X$ for $0 \leq i \leq n$ extends to a map $\Delta^n \to X$.

A good way of producing Kan complexes is as follows: Given a topological space X we may produce the simplicial set $\operatorname{Sing} X$ whose n simplices are given by continuous maps $\Delta^n \to X$, where here by Δ^n we mean the topological n simplex

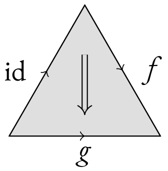
$$\Delta^n = \{(v_1, \dots, v_n) \in \mathbb{R}^n_{\geq 0} : v_1 + \cdots + v_n = 1\},$$

and face maps are given by restriction to corresponding face and degeneracy maps are given by precomposition with collapsing maps. Then one may show that $\operatorname{Sing} X$ is a Kan complex. On the other hand, given a Kan complex K we may build a topological space via *geometric realization* $|\cdot|$ which is the unique colimit preserving functor such that $|\Delta^n| = \Delta^n$, i.e. we build a topological space $|K|$ by gluing topological simplices via the manner prescribed by K. One may then should that this gives an equivalence between the theory of ∞-groupoids and topological spaces. The *homotopy hypothesis* posits that any meaningful theory of ∞-categories should have it such that ∞-groupoids correspond to topological spaces, something that is verifiable as a theorem in the simplicial set model for ∞-categories.

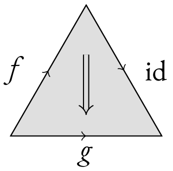
4.1.3 Homotopic maps in ∞-categories

As mentioned at the end of the previous section, we can represent ∞-categories as simplicial sets $\mathcal{C}$ such that maps from inner horns can be filled to n-simplices. However, this requirement is only for compositions to *exist*, not to be unique. It turns out that we can define a notion of homotopy in an ∞-category, and then the correct uniqueness statement is that compositions are unique up to homotopy.

Definition 4.1.9. Let $\mathcal{C}$ be an ∞-category. We say that $f \in \mathcal{C}_1$ is *left homotopic* to $g \in \mathcal{C}_1$ if there exists a 2-simplex



Similarly, we say that f is *right homotopic* to g if there exists a 2-simplex

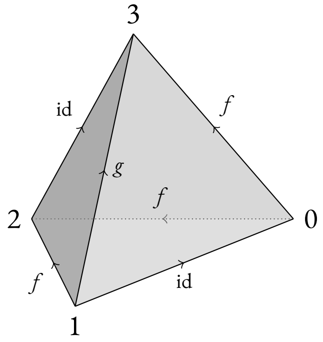


in $\mathcal{C}$.

It turns out that these a priori different definitions are in fact equivalent via our ability to compose 2-morphisms.

Proposition 4.1.10. *f is left homotopic to g if and only if f is right homotopic to g.*

Proof. Suppose we have a left homotopy from f to g given by a 2-simplex σ. Then we may consider the inner horn Λ^3_2 given by



If we fill this to a 3-simplex then extract the face opposite the vertex 2, we get a right homotopy from f to g.

A similar diagram shows the other direction. Note that we cannot use the exact some diagram as above, however, as we would be asking for a filler to an outer horn Λ^3_0. □

In light of Proposition 4.1.10, we simply say that f is *homotopic* to g if there exists a left (equiv. right) homotopy from f to g. Via similar arguments to the above, one may show the following proposition.

Proposition 4.1.11. *The relation $\simeq$ given by homotopy is an equivalence relation. Moreover, given any two composable morphisms $f, g \in \mathcal{C}_1$, any two compositions of f and g are homotopic.*

In light of this proposition, we make the following definition.

Definition 4.1.12. For $\mathcal{C}$ an ∞-category, define the ordinary category $\tau_{\leq 1}(\mathcal{C})$, also sometimes denoted $\mathrm{Ho}(\mathcal{C})$ and called the *homotopy category of* $\mathcal{C}$, to be the category whose objects are the same as those of $\mathcal{C}$ and whose morphisms are morphisms of $\mathcal{C}$ modulo homotopy. Composition is defined by any choice of composition in $\mathcal{C}$, which by Proposition 4.1.11 is independent of choice up to homotopy.

4.1.4 Relation to model categories

Suppose that we have a category with weak equivalences $(\mathcal{C}, W)$. Then we may form the localized 1-category $\mathcal{C}[W^{-1}]$ via its universal property. We can do the same procedure for ∞-categories.

Definition 4.1.13. Let $\mathcal{C}$ be an ∞-category. We say that a morphism $f : A \longrightarrow B$ in $\mathcal{C}$ is an *equivalence* if it is an isomorphism in $\mathrm{Ho}(\mathcal{C})$, i.e. there exists some $g : B \longrightarrow A$ such that $fg \simeq \mathrm{id}_B$ and $gf \simeq \mathrm{id}_A$.

Definition 4.1.14. Let $\mathcal{C}$ be an ∞-category and W a collection of morphisms in $\mathcal{C}$. We say that a functor $i : \mathcal{C} \longrightarrow \mathcal{D}$ *exhibits* $\mathcal{D}$ *as the localization of* $\mathcal{C}$ *at* W if for any ∞-category $\mathcal{E}$, precomposition with i induces a fully faithful embedding $\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ whose essential image is functors $\mathcal{C} \longrightarrow \mathcal{E}$ sending morphisms in W to equivalences.

One may show that localizations always exist and are unique up to equivalence. Moreover, we have the following not-too-hard to see result.

Theorem 4.1.15. *Let $\mathcal{C}$ be an ordinary category with weak equivalences W. Then* $\mathrm{Ho}(N(\mathcal{C})[W^{-1}]) \simeq \mathcal{C}[W^{-1}]$.

Thus ∞-categorical localizations freely turn weak equivalences into equivalences rather than isomorphisms, and we may recover the 1-categorical notion by further inverting all equivalences. One nice feature of ∞-categories in this context, however, is that limits and colimits in $N(\mathcal{C})[W^{-1}]$ correspond to taking *homotopy* limits and colimits in $\mathcal{C}[W^{-1}]$ after passing to the homotopy category.

4.1.5 Rigidification and topological categories

When referring to literature on ∞-categories, one will sometimes see them referred to as $(\infty, 1)$-categories. This notation is meant to indicate that we have n-morphisms for all $n \in \mathbb{N}$, hence the ∞, and that n-morphisms for $n > 1$ are invertible, at least up to $(n+1)$-morphism.

This may be made formal by saying that given an ∞-category $\mathcal{C}$ and objects $x, y \in \mathcal{C}$, for a suitable definition of $\mathrm{Map}_{\mathcal{C}}(x, y)$ we in fact have that $\mathrm{Map}_{\mathcal{C}}(x, y)$ is a Kan complex, i.e. an ∞-groupoid. Since the objects of $\mathrm{Map}_{\mathcal{C}}(x, y)$ are morphisms and the higher morphisms are n-morphisms $n \geq 2$ in $\mathcal{C}$, this is saying that n-morphisms in $\mathcal{C}$, $n \geq 2$, are invertible up to homotopy. Since ∞-groupoids are just topological spaces, this suggests that potentially we could have defined ∞-categories as Top-enriched ordinary categories. However, this a priori does not capture the same level of generality since a Top-enriched category has a composition law which is defined "on the nose" whereas ∞-categories only have a composition which is defined up to homotopy.

It turns out, however, that our choice of composition in an ∞-category is much more constrained then just up to homotopy. It is in fact fully determined up to all choices of higher homotopy data:

Theorem 4.1.16 ([7, Corollary 2.3.2.2]). *A simplicial set K is an ∞-category if and only if the restriction*

$$\mathrm{Fun}(\Delta^2, K) \to \mathrm{Fun}(\Lambda^2_1, K)$$

is a trivial Kan fibration.

For the purposes of our discussion, the meaning up "trivial Kan fibration" is not all too important—it simply means that we have a contractible space of choices for our composition. Using this, it is in fact possible to "rigify" any ∞-category to have an "on the nose" composition and thus truly be represented by a Top-enriched ordinary category. We won't prove this, but we will show one direction of how to produce an ∞-category from a topological (i.e. Top-enriched) category.

Definition 4.1.17. A *topological category* is an ordinary category enriched in Top, i.e. a category $\mathcal{C}$ such that each $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a topological space.

Remark 4.1.18. Via the singular complex and geometric realization, we may equivalently view a topological category as a category enriched in simplicial sets Set_{Δ}.

We now wish to define the *topological nerve* of a topological category. To do this we will define the *simplicial nerve* $N_{\Delta}(\cdot)$ of a simplicial category, and the topological nerve of a topological category $\mathcal{C}$ will simply be $N_{\Delta}(\mathrm{Sing}\,\mathcal{C})$.

Let $\mathcal{C}\mathrm{at}_{\Delta}$ denote the category of simplicial categories. Notice that we can turn every ordinary category $\mathcal{D}$ into a topological category, and hence a simplicial category, by letting the morphisms spaces be discrete topological spaces. Thus given an ordinary category $\mathcal{C}$, we may view both $[n]$ and $\mathcal{C}$ as simplicial categories. Then, one way we could have defined the nerve of $\mathcal{C}$ is via

$$N(\mathcal{C})_n = \mathrm{Hom}_{\mathcal{C}\mathrm{at}_\Delta}([n], \mathcal{C}). \tag{4.1.5}$$

However, suppose that $\mathcal{C}$ comes with a non-discrete simplicial category structure. The goal is to still use (4.1.5) as a definition, but we should be using a "sufficiently derived" Hom in the definition. That is, the homotopy theory of simplicial categories will end up modelling the theory of ∞-categories. Thus (4.1.5) should really be happening in some homotopy category of $\mathcal{C}\mathrm{at}_\Delta$ where we've inverted some notion of equivalence. When $\mathcal{C}$ is a discrete simplicial category, (4.1.5) ends up already computing a derived Hom, but in general we should take a cofibrant replacement of $[n]$ before applying Hom.

Definition 4.1.19 ([7, Definition 1.1.5.1])**.** Let J be a finite non-empty linearly ordered set. Define $\mathfrak{C}[\Delta^J]$ as follows:

- The objects of $\mathfrak{C}[\Delta^J]$ are the same as those of J
- For $i, j \in J$, then

$$\mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i,j) = \begin{cases} N(P_{i,j}) & i \leq j \\ \emptyset & j < i \end{cases}$$

  where $P_{i,j}$ is the partially ordered set $\{I \subseteq J \cap \{i, i+1, \ldots, j\} : i, j \in I\}$
- The composition $\mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i,j) \times \mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(j,k) \to \mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i,k)$ is induced by the map

$$\begin{aligned} P_{i,j} \times P_{j,k} &\longrightarrow P_{i,k} \\ (I_1, I_2) &\longmapsto I_1 \cup I_2 \end{aligned}$$

  of partially ordered sets.

This comes with a map $\mathfrak{C}[\Delta^J] \to J$ which is an equivalence of simplicial categories. The point of $\mathfrak{C}[\Delta^J]$ in relation to J is that we have forgotten compositions "on the nose" and instead remember compositions up to coherent homotopy by replacing them with chains of composable morphisms. It turns out that this assignment

$$\begin{aligned} \Delta &\longrightarrow \mathcal{C}\mathrm{at}_\Delta \\ J &\longmapsto \mathfrak{C}[\Delta^J] \end{aligned}$$

is functorial and thus defines a cosimplicial object of $\mathcal{C}\mathrm{at}_\Delta$. Thus allows us to make the following definition.

Definition 4.1.20. The *simplicial nerve* of a simplicial category $\mathcal{C}$, denoted by $N_\Delta(\mathcal{C})$ or simply

$N(\mathcal{C})$, is the simplicial set defined by

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, N_\Delta(\mathcal{C})) = \mathrm{Hom}_{\mathcal{C}\mathrm{at}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C}).$$

Remark 4.1.21. Precomposition with the equivalence $\mathfrak{C}[\Delta^n] \to [n]$ gives a bijection $\mathrm{Hom}_{\mathcal{C}\mathrm{at}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C}) \cong \mathrm{Hom}_{\mathcal{C}\mathrm{at}_\Delta}([n], \mathcal{C})$ when $\mathcal{C}$ is a discrete simplicial category, so the simplicial nerve is a generalization of the nerve of an ordinary category.

4.2 Limits and colimits

We now wish to introduce the basic notion of limits and colimits in an ∞-category. For this, we want a homotopy invariant version of the classical notion. Since homotopy invariance is already built in to the theory of ∞-categories, essentially any naive definition of limits and colimits will work. Before doing this, however, we discuss what the correct notion should be from the perspective of topological categories.

Let $\mathcal{C}$ be a topological category and suppose we have a diagram $F : \mathcal{D} \to \mathcal{C}$. Whatever $\lim_{d\in\mathcal{D}} F(d)$ is meant to be, if it exists, its functor of points should satisfy

$$\mathrm{Map}_{\mathcal{C}}(A, \lim_{d\in\mathcal{D}} F(d)) \simeq \mathrm{holim}_{d\in\mathcal{D}} \mathrm{Map}_{\mathcal{C}}(A, F(d)).$$

By the Yoneda embedding this in fact classifies the object $\lim_{d\in\mathcal{D}} F(d)$ should it exist.

4.2.1 Definition via initial and terminal objects

Definition 4.2.1. Let $\mathcal{C}$ be an ∞-category. We say that $x \in \mathcal{C}$ is an *initial object* if for every $y \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(x, y)$ is contractible. Dually, we say $x \in \mathcal{C}$ is a *terminal (or final) object* if for every $y \in \mathcal{C}$ we have that $\mathrm{Map}_{\mathcal{C}}(y, x)$ is contractible.

To define a limit of a diagram $p : K \to \mathcal{C}$ it will then suffice to define the join of two simplicial sets. Then we can consider $K^{\triangleleft} = \mathrm{pt} \star K$ which will represent freely adjoining an initial object to K. If we then define $\mathcal{C}_{/p}$ to be the full subcategory of $\mathrm{Fun}(K^{\triangleleft}, \mathcal{C})$ of functors extending p, then the limit may defined as an initial object of $\mathcal{C}_{/p}$.

Definition 4.2.2. Let S and K be simplicial sets. The *join* of S and K, denoted $S \star K$, is the simplicial set given by

$$(S \star K)(J) = \coprod_{J=I\cup I'} S(I) \times K(I')$$

where I, I' decompose J into a disjoint union such that $i < i'$ for all $i \in I, i' \in I'$. Moreover, we take the convention that $S(\emptyset) = K(\emptyset) = *$.

One should think of the join of two ∞-categories $\mathcal{C}$ and $\mathcal{C}'$ as having objects $\operatorname{ob}\mathcal{C} \amalg \operatorname{ob}\mathcal{C}'$ and morphisms

$$\operatorname{Hom}_{\mathcal{C}\star\mathcal{C}'}(A,B) = \begin{cases} \operatorname{Hom}_{\mathcal{C}}(A,B) & A,B \in \mathcal{C} \\ \operatorname{Hom}_{\mathcal{C}'}(A,B) & A,B \in \mathcal{C}' \\ * & A \in \mathcal{C}, B \in \mathcal{C}' \\ \emptyset & A \in \mathcal{C}', B \in \mathcal{C}. \end{cases}$$

Fortunately, the join of two ∞-categories remains a category.

Proposition 4.2.3 (Joyal). *If $\mathcal{C}$ and $\mathcal{C}'$ are ∞-categories, then $\mathcal{C} \star \mathcal{C}'$ is an ∞-category.*

We now may define limits and colimits in an arbitrary ∞-category.

Definition 4.2.4. For K a simplicial set, define $K^{\triangleleft} = \mathrm{pt} \star K$ and $K^{\triangleright} = K \star \mathrm{pt}$.

Given an ∞-category $\mathcal{C}$ and $p : K \to \mathcal{C}$, define $\mathcal{C}_{/p}$ and $\mathcal{C}_{p/}$ to be the full subcategories of $\operatorname{Fun}(K^{\triangleleft}, \mathcal{C})$ and $\operatorname{Fun}(K^{\triangleright}, \mathcal{C})$ respectively of functors which extend p.

Definition 4.2.5. Let $\mathcal{C}$ be an ∞-category and $p : K \to \mathcal{C}$. A *limit of p* is a final object of $\mathcal{C}_{/p}$ and a *colimit of p* is an initial object of $\mathcal{C}_{p/}$.

Remark 4.2.6. Using the simplicial model for ∞-categories, we are able to define the limit and colimit of any map $p : K \to \mathcal{C}$ of simplicial sets, even when K is not necessarily an ∞-category. This a priori appears to give a level of generality which is not present in other models for ∞-categories. However, given any simplicial set K, we may find a categorical equivalence $q : \mathcal{D} \to K$ in the Joyal model structure. We then have that the induced map $\mathcal{C}_{p/} \to \mathcal{C}_{pq/}$ is still an equivalence, so $\operatorname{colim}_K p$ exists if and only if $\operatorname{colim}_{\mathcal{D}} p \circ q$ exists, in which case the canonical comparison map $\operatorname{colim}_{\mathcal{D}} p \circ q \to \operatorname{colim}_K p$ is an equivalence. Thus we gain no loss in generality by restricting to colimits (and similarly limits) over ∞-categories. It is, however, at times convenient to allow for more general diagrams.

4.2.2 Quillen's Theorem A and B and ∞-categorical generalizations

It is often the case that various limit and colimit diagrams can be simplified to smaller ones. Take for example the case of a diagram $p : \mathcal{D} \to \mathcal{C}$ where $\mathcal{D}$ happens to have a final object $* \in \mathcal{D}$. Then we simply have that $\operatorname{colim}_{\mathcal{D}} p = p(*)$. Said differently, if $i : * \to \mathcal{D}$ is the inclusion then the induced map $\operatorname{colim}_* p \circ i \to \operatorname{colim}_{\mathcal{D}} p$ is an equivalence. The goal of cofinal morphisms is to study a general class of morphisms i for which pulling back along i preserves colimits.

Definition 4.2.7. A morphism $v : K' \to K$ of simplicial sets is *cofinal* if for every ∞-category $\mathcal{C}$ and $p : K \to \mathcal{C}$, the induced morphism $\mathcal{C}_{p/} \to \mathcal{C}_{pv/}$ is an equivalence.

It is clear from the definition of cofinality and colimits that if v is cofinal, then $\operatorname{colim}_K p$ exists if and only if $\operatorname{colim}_{K'} p \circ v$ exists in which case the canonical map $\operatorname{colim}_{K'} p \circ v \to \operatorname{colim}_K p$ is an equivalence. It turns out that this is also sufficient. In the theorem below we collect the various equivalent conditions for cofinality.

Proposition 4.2.8 ([7, Proposition 4.1.1.8]). *Let K, K' be simplicial sets and $v : K' \to K$. Then the following are equivalent:*
(i) v is cofinal
(ii) For every ∞-category $\mathcal{C}$ and $p : K \to \mathcal{C}$, if $\overline{p} : K^{\triangleright} \to \mathcal{C}$ is a colimit of p then the induced map $\overline{p}' : K'^{\triangleright} \to \mathcal{C}$ is a colimit of $p \circ v$
(iii) For any right fibration $S \to K$ of simplicial sets, the induced map $\mathrm{Map}_K(K, S) \to \mathrm{Map}_K(K', S)$ is a homotopy equivalence

Theorem 4.2.9 ([7, Theorem 4.1.3.1]). *Suppose $v : \mathcal{C} \to \mathcal{D}$ with $\mathcal{D}$ an ∞-category. Then v is cofinal if and only if $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$ is weakly contractible for every $D \in \mathcal{D}$.*

The general framework of cofinal maps in the context of ∞-categories gives an alternate point of view on Quillen's Theorem A.

Proposition 4.2.10. *A cofinal map $S \to T$ of simplicial sets is a weak homotopy equivalence.*

Proof. Let K be any other simplicial set. Then we have that the induced map

$$\begin{aligned}\mathrm{Map}_{\mathrm{Set}_\Delta}(T, K) &= \mathrm{Map}_T(T, K \times T)\\ &\to \mathrm{Map}_T(S, K \times T)\\ &= \mathrm{Map}_{\mathrm{Set}_\Delta}(S, K).\end{aligned}$$

Passing to the homotopy category of Set_Δ (in the Kan–Quillen model structure) we see that S and T represent the same functor. Thus we see that q is a homotopy equivalence. □

Corollary 4.2.11 (Quillen's Theorem A). *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories such that $B(d \downarrow F)$ is contractible for every $d \in \mathcal{D}$. Then F induces a homotopy equivalence $B\mathcal{C} \to B\mathcal{D}$.*

Proof. For a category $\mathcal{E}$, recall that $B\mathcal{E} = |N(\mathcal{E})|$. Thus the hypothesis precisely say that each $N(d \downarrow F) = N(\mathcal{C}) \times_{N(\mathcal{D})} N(\mathcal{D})_{d/}$ is weakly contractible. Thus by Theorem 4.2.9 we have that the induced map $F : N(\mathcal{C}) \to N(\mathcal{D})$ is cofinal, and hence a weak equivalence by Proposition 4.2.10. That is, we have that $B\mathcal{C} = |N(\mathcal{C})| \to |N(\mathcal{D})| = B\mathcal{D}$ is a homotopy equivalence. □

For completeness, we also include the ∞-categorical generalization of Quillen's Theorem B.

Theorem 4.2.12 ([2, Theorem 5.3]). *Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of ∞-categories such that for every morphism $X \to Y$ in $\mathcal{D}$, the induced map $Y \downarrow F \to X \downarrow F$ is a weak equivalence. Then for every $X \in \mathcal{D}$ the pullback square*

$$\begin{tikzcd}
X \downarrow F \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
* \simeq \mathcal{D}_{X/} \arrow[r] & \mathcal{D}
\end{tikzcd}$$

is homotopy Cartesian (in the Kan–Quillen model structure).

Remark 4.2.13. We have that $\mathcal{D}_{X/} \simeq *$ is weakly contractible since it has an initial object.

4.3 Stable ∞-categories

Definition 4.3.1. Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is *pointed* if it has an object which is both terminal and final, denoted by 0.

When $\mathcal{C}$ is pointed, we have that $0 \in \operatorname{Ho}(\mathcal{C})$ is still both initial and final. Thus we get a unique morphism $X \to Y$ in $\operatorname{Ho}(\mathcal{C})$ which factors through 0 and we denote this morphism by 0. Upstairs in $\mathcal{C}$, $0 : X \to Y$ is well defined up to homotopy.

Definition 4.3.2. Let $\mathcal{C}$ be a pointed ∞-category. A *triangle* in $\mathcal{C}$ is a diagram of the form

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
0 \arrow[r] & Z,
\end{tikzcd} \tag{4.3.1}$$

i.e. a functor $\Delta^1 \times \Delta^1 \to \mathcal{C}$. We call this diagram a *fiber sequence* if it is a pullback, and a *cofiber sequence* if it is a pushout.

One should be careful of what it means precisely to define a triangle. Indeed, it is not as simple as giving a "commutative diagram" of the form (4.3.1) since the notion of commutativity is not even well-defined in an ∞-category since compositions only exist up to homotopy. To state explicitly what data specifies a triangle, recall that $\Delta^1 \times \Delta^1$ has the simplicial set structure

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, "\Nearrow", "\Swarrow"'] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

with all higher dimensional simplices being degenerate. Thus the data of a triangle is

(i) A choice of zero object 0

(ii) Morphisms $f : X \to Y, g : Y \to Z, X \to 0, 0 \to Z$

(iii) A morphism $h : X \to Z$

(iv) A 2-simplex

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z
\end{tikzcd}$$

witnessing h as a composite $g \circ f$

(v) A 2-simplex

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr] & & Z \\
 & 0 \arrow[ur] &
\end{tikzcd}$$

witnessing a *null homotopy* of h.

Given a morphism $g : X \to Y$, we will write cofib(g) to denote a cofiber sequence

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\
0 \arrow[r] & Z
\end{tikzcd} \tag{4.3.2}$$

and similarly we will write fib(g) to denote a fiber sequence arising from g.

Remark 4.3.3. It may appear at first that the notation cofib(g) and fib(g) is poor since for each g we are making a choice of triangle from a contractible space of such choices. It turns out, however, that we may assume this assignment is functorial. Indeed, there is on a contractible space of zero objects, followed by a contractible space of maps $X \to 0$ and then finally a contractible space of colimits forming the pushout. If we let CoFib denote the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ corresponding to cofiber sequences, then the projection $\mathrm{CoFib} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ onto the top arrow (labelled g in (4.3.2)) is a Kan fibration which has fibers either empty or Kan complexes corresponding to whether a cofiber of $g \in \mathrm{Fun}(\Delta^1, \mathcal{C})$ exists. In particular, if all cofibers exist in $\mathcal{C}$, then $\mathrm{CoFib} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ is a trivial Kan fibration and hence has a section $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{CoFib}$. In particular, when all cofibers exist we may make a functorial assignment $g \mapsto \mathrm{cofib}(g)$. The same is true for fiber sequences.

Definition 4.3.4. An ∞-category $\mathcal{C}$ is *stable* if

(i) $\mathcal{C}$ is pointed

(ii) all fibers and cofibers exist in $\mathcal{C}$

(iii) a triangle is a cofiber sequence if and only if it is a fiber sequence.

One should think of stable ∞-categories as the ∞-categorical generalization of triangulated categories. Formally, if $\mathcal{C}$ is an ∞-category, then one may show that the homotopy category $\mathrm{Ho}(\mathcal{C})$ is a triangulated category. The benefit of stable ∞-categories is that being stable is a *property* of an ∞-category, whereas being triangulated requires supplementing the data of distinguished triangles.

We now discuss how the triangulated structure of $\mathrm{Ho}(\mathcal{C})$ appears on the ∞-categorical level. In particular, we first discuss where the shifts arise from. For this, let $\mathcal{C}^{\Sigma}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ spanned by cofiber sequences of the form

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & 0 \arrow[d] \\
0' \arrow[r] & Y
\end{tikzcd}$$

where 0 and $0'$ are zero objects. We then have a forgetful map $\mathcal{C}^{\Sigma} \to \mathcal{C}$ given by evaluating at the initial vertex. Provided all such pushouts exists, $\mathcal{C}^{\Sigma} \to \mathcal{C}$ is a trivial Kan fibration and hence has a section $\mathcal{C} \to \mathcal{C}^{\Sigma}$. Post-composing with projection to the terminal vertex we get a functor $\Sigma : \mathcal{C} \to \mathcal{C}$ such that for every X we have a cofiber sequence

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & 0 \arrow[d] \\
0' \arrow[r] & \Sigma X.
\end{tikzcd}$$

Dually one may consider fiber sequences and produce a functor $\Omega : \mathcal{C} \to \mathcal{C}$ sitting in a fiber sequence

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] & 0 \arrow[d] \\
0' \arrow[r] & X.
\end{tikzcd}$$

If $\mathcal{C}$ is stable then $\mathcal{C}^{\Sigma} = \mathcal{C}^{\Omega}$ and Σ, Ω are inverse equivalences. However, if $\mathcal{C}$ is just pointed then Σ is only necessarily left adjoint to Ω.

If $\mathcal{C}$ is stable, then we may define *shift* functors $X \mapsto X[n]$ for each $n \in \mathbb{Z}$ via

$$(-)[n] = \begin{cases} \Sigma^n & \text{if } n \geq 0 \\ \Omega^{-n} & \text{if } n < 0. \end{cases}$$

The truncation of this shift functor defines the shift functor on $\mathrm{Ho}(\mathcal{C})$ and we essentially call a triangle

$$X \to Y \to Z \to X[1]$$

in $\mathrm{Ho}(\mathcal{C})$ distinguished if it arises from a diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\
0' \arrow[r] & Z \arrow[r] & W
\end{tikzcd}$$

where both the smaller squares (and hence the outer square too) are pushouts.

4.4 Derived world revisited

4.4.1 *t*-structures

Suppose we have an abelian category $\mathcal{A}$. We can then consider the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$. From $\mathrm{Ch}(\mathcal{A})$ we can in fact, after inverting quasi-equivalences, recover $\mathcal{A}$ as those chain complexes $C_\bullet$ such that $H_n(C_\bullet) = 0$ for all $n \neq 0$. The goal of t-structures is to generalize the idea of being "homologically concentrated in degree ≥ 0" to any stable ∞-category.

Definition 4.4.1. Let $\mathcal{T}$ be a triangulated category. A *t-structure* on $\mathcal{T}$ is the data of two full subcategories $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ such that
(i) if $X \in \mathcal{T}_{\geq 0}$ and $Y \in \mathcal{T}_{\leq 0}$, then $\mathrm{Hom}(X, Y[-1]) = 0$
(ii) $\mathcal{T}_{\leq 0}[-1] \subseteq \mathcal{T}_{\leq 0}$ and $\mathcal{T}_{\geq 0}[1] \subseteq \mathcal{T}_{\geq 0}$
(iii) For any $X \in \mathcal{T}$, there exists a fiber sequence $Y \to X \to Y'$ with $Y \in \mathcal{T}_{\geq 0}$ and $Y' \in \mathcal{T}_{\leq 0}[-1]$.

In this context, we will write $\mathcal{T}_{\geq n}$ to denote $\mathcal{T}_{\geq 0}[n]$ and $\mathcal{T}_{\leq n}$ to denote $\mathcal{T}_{\leq 0}[n]$. Note that $\mathcal{T}_{\geq 0}$ entirely determines $\mathcal{T}_{\leq -1}$ and hence $\mathcal{T}_{\leq 0} = \mathcal{T}_{\leq -1}[1]$. Indeed, an object Y belongs to $\mathcal{T}_{\leq -1}$ if and only if $\mathrm{Hom}(X, Y) = 0$ for all $X \in \mathcal{T}_{\geq 0}$. Condition (i) makes this as a necessary condition and (iii) shows that it is sufficient.

Definition 4.4.2. Let $\mathcal{C}$ be a stable ∞-category. A *t-structure* on $\mathcal{C}$ is a t-structure on $\mathrm{Ho}(\mathcal{C})$.

Given a t-structure on $\mathcal{C}$, we write $\mathcal{C}_{\geq 0}$ for the full subcategory of objects of $\mathcal{C}$ which lie in $\mathrm{Ho}(\mathcal{C})_{\geq 0}$ after passing to the homotopy category. Similarly for $\mathcal{C}_{\leq 0}$. An important fact about t-structures is that $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ is a reflexive subcategory, and dually $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ has a right adjoint.

Proposition 4.4.3 ([8, Proposition 1.2.1.5])**.** *The inclusion $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ has a left adjoint. Dually, the inclusion $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ has a right adjoint.*

As a consequence of this, we write $\tau_{\leq n} : \mathcal{C} \to \mathcal{C}_{\leq n}$ for the left adjoint to $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$, and $\tau_{\geq n} : \mathcal{C} \to \mathcal{C}_{\geq n}$ for the right adjoint to $\mathcal{C}_{\geq n} \subseteq \mathcal{C}$. One may show that $\tau_{\geq n} \circ \tau_{\leq m} \simeq \tau_{\leq m} \circ \tau_{\geq n}$ so may unambiguously truncate objects into $\mathcal{C}_{\leq m} \cap \mathcal{C}_{\geq n}$. Using this, we make the following definition.

Definition 4.4.4. Let $\mathcal{C}$ be a stable ∞-category with t-structure. The *heart* of $\mathcal{C}$, denoted $\mathcal{C}^{\heartsuit}$, is the full subcategory $\mathcal{C}_{\leq 0} \cap \mathcal{C}_{\geq 0}$. Define $\pi_0 : \mathcal{C} \longrightarrow \mathcal{C}^{\heartsuit}$ to be $\tau_{\geq 0} \circ \tau_{\leq 0} \simeq \tau_{\leq 0} \circ \tau_{\geq 0}$ and $\pi_n : \mathcal{C} \longrightarrow \mathcal{C}^{\heartsuit}$ to be $\pi_0 \circ (-)[-n]$, i.e. $\pi_n X = \pi_0(X[-n])$.

Note that $\mathcal{C}^{\heartsuit}$ is equivalent to an ordinary category, i.e. $\mathcal{C}^{\heartsuit} \simeq N(\mathrm{Ho}(\mathcal{C}^{\heartsuit}))$. This is because for any two $X, Y \in \mathcal{C}^{\heartsuit}$, we have that for all $n > 0$

$$\begin{aligned} \pi_n \operatorname{Map}(X, Y) &= \pi_0 \Omega^n \operatorname{Map}(X, Y) \\ &= \pi_0 \operatorname{Map}(X, Y[-n]) \\ &= 0. \end{aligned}$$

Hence the mapping spaces are discrete for any two objects in $\mathcal{C}^{\heartsuit}$. It turns out that the stability of $\mathcal{C}$ also implies that $\mathcal{C}^{\heartsuit}$ is abelian, so one may think of $\mathcal{C}$ as a derived enhancement of the ordinary abelian category $\mathcal{C}^{\heartsuit}$.

4.4.2 ∞-fication of classical derived categories

In this section we discuss how one can view the usual derived categories in homological algebra as ∞-categories.

Let $\mathcal{A}$ be an abelian category. In homological algebra, one may form the right bounded derived category $\mathcal{D}^-(\mathcal{A})$ as follows:

(i) Form the quotient category $K^-(\mathcal{A}) = \mathrm{Ch}^-(\mathcal{A})/\simeq$ given by taking all right bounded chain complexes in $\mathcal{A}$ and identifying all chain homotopic maps

(ii) Form $\mathcal{D}^-(\mathcal{A}) = K^-(\mathcal{A})[W^{-1}]$ by further inverting the set W of all quasi-isomorphisms.

It is a simple exercise to show that, up to equivalence of categories, step (i) is unnecessary. Indeed, inverting all quasi-isomorphisms has the consequence of identifying maps which induce the same map on homology, and homotopic maps indeed induce the same map on homology. Thus simply inverting quasi-isomorphisms already has the effect of identifying homotopic maps. However, in the classical story, it is convenient to first do (i) as it is easy to give $K^-(\mathcal{A})$ a triangulated structure which is then inherited by $K^-(\mathcal{A})[W^{-1}]$, whereas directly giving a triangulated structure to $\mathrm{Ch}^-(\mathcal{A})[W^{-1}]$ is difficult.

Our goal in this section is to build a stable ∞-category $\mathcal{D}^-(\mathcal{A})$ whose homotopy category is the classicial right bounded derived category with its natural t-structure. This may be achieved in two steps. First, to replicate $K^-(\mathcal{A})$, we need to build an ∞-categorical model of $\mathrm{Ch}^-(\mathcal{A})$ whose 2-morphisms are chain homotopies, whose 3-morphisms are homotopies between homotopies, and so on, at which point passing to the homotopy category will have the effect of identifying homotopic maps. After this, we need to invert quasi-isomorphisms in the ∞-categorical sense so as to make them equivalences.

We first take on the task of encoding our notion of chain homotopies into an ∞-category whose objects are those of $\mathrm{Ch}^-(\mathcal{A})$.

Definition 4.4.5. Let k be a commutative ring. A *differential graded category over k* is an (ordinary) category $\mathcal{C}$ enriched over $\mathrm{Ch}(k)$.

Now, the inclusion $\mathrm{Ch}_{\geq 0}(k) \subseteq \mathrm{Ch}(k)$ is a reflective subcategory with localization map $\tau_{\geq 0} : \mathrm{Ch}(k) \to \mathrm{Ch}_{\geq 0}(k)$ given by sending

$$\cdots \longrightarrow A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A_{-1} \xrightarrow{d_{-1}} \cdots$$

to

$$\cdots \longrightarrow A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} \ker(d_0) \longrightarrow 0.$$

This localization map is monoidal, so we may view every differential graded category over k as a category enriched over $\mathrm{Ch}_{\geq 0}(k)$, or by forgetting the k-module structure as enriched over $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$. We will later see in §4.4.3 that there is a lax monoidal functor $\mathrm{DK} : \mathrm{Ch}_{\geq 0}(\mathbb{Z}) \to \mathrm{Set}_\Delta$ given by the Dold–Kan correspondence. As such, every differential graded category over k gives rise to a simplicial, or equivalently topological, category. For later reference, we label the composite functor

$$K : \mathrm{Ch}(\mathbb{Z}) \xrightarrow{\tau_{\geq 0}} \mathrm{Ch}_{\geq 0}(\mathbb{Z}) \xrightarrow{\mathrm{DK}} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Set}_\Delta \tag{4.4.1}$$

as K.

Definition 4.4.6. Let $\mathcal{C}$ be a differential graded category (over k). The *differential graded nerve* of $\mathcal{C}$, denoted $N_{\mathrm{dg}}(\mathcal{C})$, is the simplicial (or topological) nerve of the associated simplicial category to $\mathcal{C}$ as described above.

We now use this theory to construct an ∞-categorical enrichment of $\mathrm{Ch}^-(\mathcal{A})$ which includes chain homotopies. Recall that $\mathrm{Ch}(\mathcal{A})$ naturally comes with the structure of a differential graded category as follows: Given two chain complexes $A_\bullet$ and $B_\bullet$, we need to build a mapping chain complex $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)$. For this, let

$$\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)_n = \prod_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(A_p, B_{p+n})$$

and we define a differential $d : \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)_n \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)_{n-1}$ by

$$(df)_p(x) = d(f_p(x)) - (-1)^n f_{p-1}(dx).$$

Note that the $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)_n$ encodes arbitrary collections of maps $(A_p \to B_{p+n})_p$ with no additional assumptions that, e.g., they commute with the differentials. However, the differentials encode compatibility with the differentials of $A_\bullet$ and $B_\bullet$. Indeed, elements of

$$\ker(d : \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)_0 \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)_{-1})$$

are precisely the chain maps $A_\bullet \to B_\bullet$ and elements of

$$\ker(d : \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)_1 \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)_0)$$

are precisely chain maps $A_\bullet \to B_{\bullet+1}$. Moreover,

$$H_0(\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)_\bullet) = \ker(d_0)/\mathrm{im}(d_1)$$

is precisely the additive group of chain maps $A_\bullet \to B_\bullet$ modulo chain homotopy. More generally, $H_n(\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)_\bullet)$ represents chain homotopy classes of maps $A_\bullet \to B_{\bullet+n}$. This structure turns $\mathrm{Ch}(\mathcal{A})$ into a differential graded category, and hence also turns the full subcategories $\mathrm{Ch}^-(\mathcal{A})$ and $\mathrm{Ch}^+(\mathcal{A})$ of right and left bounded, respectively, chain complexes into differential graded categories.

Definition 4.4.7. Define $K(\mathcal{A}) = N_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$, $K^-(\mathcal{A}) = N_{\mathrm{dg}}(\mathrm{Ch}^-(\mathcal{A}))$ and $K^+(\mathcal{A}) = N_{\mathrm{dg}}(\mathrm{Ch}^+(\mathcal{A}))$.

Remark 4.4.8. This notation is consistent with the classical notation as $\mathrm{Ho}(K(\mathcal{A})) = \mathrm{Ch}(\mathcal{A})/\simeq$ is the quotient category of $\mathrm{Ch}(\mathcal{A})$ modulo chain homotopy, per the discussion above. Similarly for $K^-(\mathcal{A})$ and $K^+(\mathcal{A})$.

Proposition 4.4.9. *The category $K(\mathcal{A})$ has all finite limits and colimits.*

Proof. It suffices to exhibit an initial and terminal object, as well as show that pullbacks and pushouts exist. For initial and final objects, we have that the zero complex is both initial and final. Indeed, given a chain complex $A_\bullet$ we have that

$$\mathrm{Map}_{K(\mathcal{A})}(0, A_\bullet) = K(\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(0, A_\bullet)_\bullet) = K(0) \simeq \mathrm{pt}$$

is contractible (recall K is the composite (4.4.1)). Similarly, $\mathrm{Map}_{K(\mathcal{A})}(A_\bullet, 0)$ is contractible.

We now discuss how to construct pushouts. Suppose we have $f : C \to D, g : C \to D'$. We may form a new chain complex $C(f, g)_\bullet$ with

$$C(f, g)_n = D_n \oplus D'_n \oplus C_{n-1}$$

with differential

$$\partial(d, d', c) = (\partial d + f(c), \partial d' - g(c), \partial c).$$

Morally, one should think of this as a pair of elements in D and D', along with an element of C witnessing a homotopy between them. One may then build a diagram

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "g"] & D \arrow[d] \arrow[dl, Rightarrow, shorten=4mm] \\
D' \arrow[r] & C(f, g)
\end{tikzcd}$$

which commutes up to homotopy and realizes $C(f, g)$ as the pushout of $C \to D$, $C \to D'$. Similarly for pullbacks. □

This proof also shows that $K(\mathcal{A})$ is pointed and upon working out an explicit model for pullbacks, one learns that a square in $K(\mathcal{A})$ is a pullback if and only if it is a pushout. As such, we get the following corollary.

Proposition 4.4.10. $K(\mathcal{A})$ *is stable (similarly for* $K^-(\mathcal{A})$ *and* $K^+(\mathcal{A})$*) and the induced triangulated structure on* $\mathrm{Ho}(K(\mathcal{A}))$ *agrees with the classical one.*

We have thus succeeded in the first portion of our goal which is to realize $\mathrm{Ch}(\mathcal{A})$ as a stable ∞-category which remembers chain homotopies as higher morphisms. Now we need to invert quasi-isomorphisms. This may be done in two ways: Either we take the Kan–Dwyer localization (c.f. §4.1.4) at the collection W of quasi-isomorphisms, i.e. $K(\mathcal{A})[W^{-1}]$, or we can "quotient" by the subcategory of all complexes with zero homology. We can make this sort of quotienting formally in the following manner.

Definition 4.4.11. Let $\mathcal{C}$ be a stable ∞-category. A *stable subcategory* is a stable full subcategory $\mathcal{D} \subseteq \mathcal{C}$ such that the inclusion is exact (i.e. preserves finite limits and colimits).

Definition 4.4.12. Let $\mathcal{C}$ be a stable ∞-category and $\mathcal{D} \subseteq \mathcal{C}$ a stable subcategory. Then the *Verdier quotient* $\mathcal{C}/\mathcal{D}$ is defined as the localization $\mathcal{C}[W^{-1}]$ where W is the collection of all morphisms whose cone lies in $\mathcal{D}$.

It a basic fact of Verdier quotients that the quotient is still stable.

Proposition 4.4.13. *If $\mathcal{C}$ is a stable ∞-category and $\mathcal{D} \subseteq \mathcal{C}$ is a stable subcategory, then $\mathcal{C}/\mathcal{D}$ is stable.*

Proof. See [9, Theorem I.3.3] for details. □

We are thus ready to make our definitions. Note that the subcategory $N(\mathcal{A}) \subseteq K(\mathcal{A})$ of all chain complexes with zero homology is closed under shifts, fibers and cofibers, hence stable with exact inclusion.

Definition 4.4.14. Define $\mathcal{D}(\mathcal{A}) = K(\mathcal{A})/N(\mathcal{A})$, $\mathcal{D}^-(\mathcal{A}) = K^-(\mathcal{A})/(N(\mathcal{A}) \cap K^-(\mathcal{A}))$ and $\mathcal{D}^+(\mathcal{A}) = K^+(\mathcal{A})/(N(\mathcal{A}) \cap K^+(\mathcal{A}))$.

Combining all the results up to this point, we conclude with the following theorem.

Theorem 4.4.15. *$\mathcal{D}(\mathcal{A})$ is a stable ∞-category whose homotopy category agrees with the classical unbounded, triangulated derived category. The induced classical t-structure on $\mathcal{D}(\mathcal{A})$ is such that $\mathcal{D}(\mathcal{A})^\heartsuit \simeq \mathcal{A}$.*

The analogous results hold for $\mathcal{D}^-(\mathcal{A})$ and $\mathcal{D}^+(\mathcal{A})$.

We conclude this section with a discussion of the alternative approaches for constructing $\mathcal{D}(\mathcal{A})$. Just like in the classical picture, we could of immediately started with $\mathrm{Ch}(\mathcal{A})$ and inverted quasi-isomorphisms—making all quasi-isomorphisms equivalences already has the affect of accounting for chain homotopies.

Theorem 4.4.16. *There is a canonical equivalence $N(\mathrm{Ch}(\mathcal{A}))[W^{-1}] \simeq \mathcal{D}(\mathcal{A})$, where W is the collection of all quasi-isomorphisms.*

Proof. See [8, Theorem 1.3.4.4]. □

Similarly, if $\mathcal{A}$ has enough projectives or injectives, we get an alternative presentation of $\mathcal{D}^-(\mathcal{A})$ and $\mathcal{D}^+(\mathcal{A})$. One should think of this as being a consequence of the injective and projective model structures on $\mathrm{Ch}^-(\mathcal{A})$ and $\mathrm{Ch}^+(\mathcal{A})$ which allow us to view the localized category as a quotient category of the subcategories of fibrant-cofibrant objects.

Theorem 4.4.17. *(i) Suppose $\mathcal{A}$ has enough projectives, then $\mathcal{D}^-(\mathcal{A}) \simeq N_{\mathrm{dg}}(\mathrm{Ch}^-(\mathcal{A}_{\mathrm{proj}}))$ where $\mathcal{A}_{\mathrm{proj}} \subseteq \mathcal{A}$ is the full subcategory of projective objects.*

(ii) Suppose $\mathcal{A}$ has enough injectives, then $\mathcal{D}^+(\mathcal{A}) \simeq N_{\mathrm{dg}}(\mathrm{Ch}^+(\mathcal{A}_{\mathrm{inj}}))$ where $\mathcal{A}_{\mathrm{inj}} \subseteq \mathcal{A}$ is the full subcategory of injective objects.

4.4.3 The Dold–Kan correspondence

In classical homological algebra, the way we "derive" or "triangulate" an abelian category $\mathcal{A}$ is by considering chain complexes valued in $\mathcal{A}$. However, when $\mathcal{C}$ is some not-necessarily-abelian category, we need a different approach to "triangulate" $\mathcal{C}$. The modern approach towards doing this is by using simplicial techniques, and Dold–Kan gives a correspondence between this approach and the chain complex approach in the abelian case.

The motivating example of this correspondence is from topology. Let X be a topological space. Then we may construct the *singular simplicial set* $\operatorname{Sing} X$ by letting

$$(\operatorname{Sing} X)_n = \operatorname{Hom}_{\mathrm{Top}}(\Delta^n, X),$$

where here $\Delta^n = \{(v_1, \dots, v_n) \in \mathbb{R}^n_{\geq 0} : v_1 + \cdots + v_n = 1\}$ is the topological n-simplex, and letting the face and degeneracy maps being given by restriction to faces and precomposition with collapsing maps $\Delta^{n+1} \to \Delta^n$. One may then form the free simplicial abelian group $\mathbb{Z}[\operatorname{Sing} X]$ by letting

$$\mathbb{Z}[\operatorname{Sing} X]_n = \mathbb{Z}[(\operatorname{Sing} X)_n].$$

Finally, from $\mathbb{Z}[\operatorname{Sing} X]$ we may build a natural chain complex

$$\cdots \longrightarrow \mathbb{Z}[\operatorname{Sing} X]_2 \xrightarrow{\partial_2} \mathbb{Z}[\operatorname{Sing} X]_1 \xrightarrow{\partial_1} \mathbb{Z}[\operatorname{Sing} X]_0 \longrightarrow 0 \tag{4.4.2}$$

where

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i$$

is the alternating sum of the face maps in $\mathbb{Z}[\operatorname{Sing} X]$. (4.4.2) is typically referred to as the *singular chain complex* associated to X, and its homology computes the homology of X.

It turns out that this procedure is reversible and induces an equivalence between chain complexes concentrated in non-negative degrees and simplicial abelian groups. In fact, the correspondence works for chain complexes valued in any additive category $\mathcal{A}$ giving a comparison with $\operatorname{Fun}(\Delta^{\mathrm{op}}, \mathcal{A})$, i.e. simplicial objects in $\mathcal{A}$.

Definition 4.4.18 ([8, Construction 1.2.3.5])**.** Let $\mathcal{A}$ be an additive category. The *Dold–Kan construction* $\mathrm{DK} : \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \operatorname{Fun}(\Delta^{\mathrm{op}}, \mathcal{A})$ is defined as follows: Let $(A_\bullet, d)$ be a chain complex valued in $\mathcal{A}$.

(i) For each n,

$$\mathrm{DK}(A_\bullet)_n = \bigoplus_{\alpha : [n] \twoheadrightarrow [k]} A_k$$

where the direct sum ranges over all surjective maps $\alpha : [n] \to [k]$ in Δ.

(ii) Given $\beta : [n'] \to [n]$ we define

$$\beta^* : \mathrm{DK}(A_\bullet)_n = \bigoplus_{\alpha:[n]\twoheadrightarrow[k]} A_k \to \bigoplus_{\alpha':[n']\twoheadrightarrow[k']} A_{k'} = \mathrm{DK}(A_\bullet)_{n'}$$

by a matrix of maps $(f_{\alpha,\alpha'})$ where

- (a) $f_{\alpha,\alpha'} = \mathrm{id}$ if $\alpha' = \alpha\beta$
- (b) $f_{\alpha,\alpha'} = d$ if $k' = k - 1$ and the diagram

$$\begin{tikzcd}
{[n']} \arrow[rr, "\beta"] \arrow[d, "\alpha'"] & & {[n]} \arrow[d, "\alpha"] \\
{[k']} \arrow[r, "\cong"] & \{1, \dots, k\} \arrow[r] & {[k]}
\end{tikzcd}$$

  commutes
- (c) $f_{\alpha,\alpha'} = 0$ otherwise.

We then have the following result.

Theorem 4.4.19 (Dold–Kan Correspondence)**.** *Let $\mathcal{A}$ be an additive category. Then the Dold–Kan functor*

$$\mathrm{DK} : \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A})$$

is fully faithful. If $\mathcal{A}$ is idempotent complete, then DK *is an equivalence.*

Moreover, when $\mathcal{A} = $ Ab*,* DK *induces an equivalence of homotopy categories as well.*

Rather than prove this, we give an explicit inverse equivalence in the case of $\mathcal{A}$ abelian.

Definition 4.4.20. Let $\mathcal{A}$ be an additive category and $A_\bullet$ a simplicial object in $\mathcal{A}$. The *unnormalized chain complex of $A_\bullet$*, denoted $C_*(A)$, is the chain complex

$$\cdots \longrightarrow A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \longrightarrow 0$$

where

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i$$

is the alternating sum of the face maps of $A_\bullet$.

Up to quasi-isomorphism, C_* is the inverse equivalence to DK. However, when $\mathcal{A}$ is abelian we can get an adjoint equivalence on the nose by removing some "fluff" from $C_*(A)$.

Definition 4.4.21. Let $\mathcal{A}$ be an abelian category and $A_\bullet$ a simplicial object in $\mathcal{A}$. The *normalized chain complex of $A_\bullet$*, denoted $N_*(A)$, is the chain complex given by

$$N_n(A) = \bigcap_{1 \le i \le n} \ker d_i = \ker \Big(\bigoplus_{1 \le i \le n} d_i : A_n \to \bigoplus_{1 \le i \le n} A_{n-1} \Big)$$

with differentials d_0, i.e.

$$\cdots \longrightarrow N_2(A) \xrightarrow{d_0} N_1(A) \xrightarrow{d_0} N_0(A) \longrightarrow 0.$$

It is immediate from definition that there is a monomorphism $N_*(A) \to C_*(A)$, but it is not clear that this induces an isomorphism on homology. To show this we take a detour to quantify exact which "fluff" we remove my passing to the normalized chain complex.

Definition 4.4.22. Let $\mathcal{A}$ be an abelian category and $A_\bullet$ a simplicial object in $\mathcal{A}$. Define the *chain complex of degenerate simplices of $A_\bullet$*, denoted $D_*(A)$, to be

$$D_n(A) = \bigcup_{0 \le i \le n-1} \operatorname{im} s_i = \operatorname{im} \Big(\bigoplus_{0 \le i \le n-1} s_i : \bigoplus_{0 \le i \le n-1} A_{n-1} \to A_n \Big),$$

where here the s_i are the degeneracy maps of $A_\bullet$, with differentials alternating sums of face maps.

It is an easy check using the face and degeneracy map relations that the differential on $D_*(A)$ is well-defined, i.e. carries degenerate simplices to degenerate simplicies. From definition we also immediately see that $D_*(A) \subseteq C_*(A)$.

Proposition 4.4.23. *The composite map*

$$N_*(A) \hookrightarrow C_*(A) \twoheadrightarrow C_*(A)/D_*(A)$$

is an isomorphism.

Proof. See [6, Theorem III.2.1]. To prove this, one fixes $n \in \mathbb{N}$ and for each $1 \le j \le n$ defines

$$N^j A_n = \bigcap_{j \le i \le n} \ker d_i$$

and

$$D^j A_n = \bigcup_{j-1 \le i \le n-1} \operatorname{im} s_i.$$

Then one shows that the induced map

$$N^j A_n \to A_n \to A_n / D^j A_n$$

is an isomorphism for all j by downward induction. The case $j = 1$ is then the claimed result. $\square$

Corollary 4.4.24. *The inclusions* $N_*(A) \hookrightarrow C_*(A)$ *and* $D_*(A) \hookrightarrow C_*(A)$ *induce an isomorphism* $C_*(A) \cong N_*(A) \oplus D_*(A)$.

Proof. We have the following split short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & D_*(A) \arrow[r] & C_*(A) \arrow[r] & C_*(A)/D_*(A) \arrow[r] \arrow[d, "\cong"] & 0 \\
 & & & N_*(A) \arrow[ul] &
\end{tikzcd}$$

from which the result follows. $\square$

Thus the normalized chain complex is simply a complement to the degenerate simplices inside $C_*(A)$, i.e. the "fluff" we threw out was precisely choosing a canonical choice of simplex representative for each class modulo degenerate simplices.

Theorem 4.4.25. *The complex* $D_*(A)$ *is contractible. In particular, the inclusion* $N_*(A) \hookrightarrow C_*(A)$ *is a homotopy equivalence (and thus a quasi-isomorphism).*

Proof. See [6]. $\square$

We now state the main part of the correspondence for abelian categories.

Theorem 4.4.26. *Let* $\mathcal{A}$ *be an abelian category. There is a canonical isomorphism* $\mathrm{id}_{\mathrm{Ch}_{\geq 0}(\mathcal{A})} \cong N_* \circ \mathrm{DK}$ *which exhibits* N_* *as a right adjoint to* DK.

One final remark we make is that, in the case of $\mathcal{A} = \mathrm{Ab}$, the category of abelian groups, every simplicial abelian group $A_\bullet$ is a Kan complex with a canonical base point $0 \in A_\bullet$. Thus we may define $\pi_n A = \pi_n(|A_\bullet|, 0)$ and the abelian group structure on $\pi_n A$ induced by the group structure of A agrees with the usual group structure of $\pi_n A$. Chasing through the definitions, one also finds that $H_n(N_*(A)) = \pi_n A$. This has the following application:

Example 4.4.27. Let A be an abelian group. Then one has the following construction for the Eilenberg–MacLane space $K(A, n)$:

$$K(A, n) = |\mathrm{DK}_\bullet(A[-n])|.$$

Indeed, we have a Quillen equivalence between $|\cdot|$ and Sing, so

$$\begin{aligned}
[X, |\mathrm{DK}_\bullet(A[-n])|] &\cong [\mathrm{Sing}\, X, \mathrm{DK}_\bullet(A[-n])] \\
&\cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Ab}))}(\mathbb{Z}[\mathrm{Sing}\, X], \mathrm{DK}_\bullet(A[-n])) \\
&\cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Ch}_{\geq 0}(\mathrm{Ab}))}(N_*(\mathbb{Z}[\mathrm{Sing}\, X]), A[-n]) \\
&\cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Ch}_{\geq 0}(\mathrm{Ab}))}(C_*(\mathbb{Z}[\mathrm{Sing}\, X]), A[-n]) \\
&\cong H^n(X, A).
\end{aligned}$$

This both shows that $H^n(-, A)$ is a homotopy invariant and that it is representable in the homotopy category of spaces. Moreover, we have that

$$\pi_m |\mathrm{DK}_\bullet(A[-n])| = H_m(A[-n]) = \begin{cases} A & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases}$$

Call this explicit model $K(A, n)$. Moreover, by Hurewicz and the UCT we see that $H^n(K(A, n), A) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(K(A, n)), A) \cong \mathrm{Hom}_{\mathbb{Z}}(A, A)$. Under this isomorphism, one checks that the universal element $F \in H^n(K(A, n), A)$ corresponds to $\mathrm{id} : A \to A$.

Now suppose that X is another path connected space with

$$\pi_m X = \begin{cases} A & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases} \tag{4.4.3}$$

Then similarly by Hurewicz theorem and UCT, we have that $H^n(X, A) \cong \mathrm{Hom}_{\mathbb{Z}}(A, A)$. Thus $\mathrm{id} : A \to A \in H^n(X, A)$ induces a map $\varphi : X \to K(A, n)$. Now, let $x \in H_n(X)$. Then we have that

$$\begin{aligned}
\varphi_*(x) &= F(\varphi_*(x)) \\
&= (\varphi^* F)(x) \\
&= x
\end{aligned}$$

by construction, so φ induces the identity $H_n(X) \to H_n(K(A, n))$. Since there is only one non-zero homotopy group we only need to check that φ induces an isomorphism on π_n, but this follows from the above and Hurewicz. Thus we have also shown that $K(A, n)$ is uniquely determined up to weak equivalence by (4.4.3).

4.4.4 Monoidal Dold–Kan and dg-models

One may show that the Dold–Kan correspondence is lax monoidal and thus induces an adjunction between algebra objects in the respective categories. In characteristic 0, this induces a Quillen equivalence between commutative differential graded algebras and simplicial algebras. This is important as it allows one to do derived algebraic geometric in characteristic zero using differential graded algebras, which are typically easier to work with.

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