method of lines.

discretize space first, leaving time continuous

e.g. 2nd order space: $\frac{\partial u_j}{\partial t} = D_x^+ D_x^- u_j$, $\frac{\partial u}{\partial t} = Bu$, $B = D_x^+ D_x^-$

then discretize this ODE in time using a scheme from 228A.

e.g. trapezoidal rule: $\begin{cases} u' = f(u) \\ U^{n+1} = U^n + \frac{k}{2}\left[f(U^n) + f(U^{n+1}) \right] \end{cases}$

in our case, $f(u) = Bu$ is linear, but you can also use this approach for nonlinear PDE's.

The trapezoidal rule is a Runge-Kutta method with Butcher array

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

general Runge Kutta method (s-stages, solving the ODE $u' = f(u)$)

$$\ell_i = f\left(U^n + k \sum_{j=1}^{s} a_{ij} \ell_j \right) \qquad 1 \le i \le s$$

$$U^{n+1} = U^n + k \sum_{j=1}^{s} b_j \ell_j$$

the ℓ_i are called stage derivatives

truncation error: $\tau^n = \frac{1}{k}\left[u(t_n + k) - u(t_n) - k \sum_{j=1}^{s} b_j \ell_j(k) \right]$

$u(t)$ = exact sol'n, scheme initialized with $u(t_n)$

then we Taylor expand the solution and the scheme

$$u(t_n+k) = u(t_n) + k\,u'(t_n) + \cdots + \frac{k^{p+1}}{(p+1)!}u^{(p+1)}(t_n) + O(k^{p+2})$$

$$\ell_i(k) = f\Big(u^n + k\sum_{j=1}^{s} a_{ij}\,\ell_j(k)\Big)$$

$$= \ell_i(0) + k\,\ell_i'(0) + \cdots + \frac{k^p}{p!}\ell_i^{(p)}(0) + O(k^{p+1})$$

↑
see 228A notes

result:

$$\tau^n = \frac{1}{k}\Big[\Big(u' - \sum_j b_j \ell_j\Big)k + \Big(\tfrac{1}{2}u'' - \sum_j b_j \ell_j'\Big)k^2 + \cdots + \Big(\frac{1}{(p+1)!}u^{(p+1)} - \frac{1}{p!}\sum_j b_j \ell_j^{(p)}\Big)k^{p+1}\Big] + O(k^{p+1})$$

tool for evaluating these terms: labeled trees

formulas:

$$u^{(q)}(t_n) = \sum_{\phi\in T_q} \alpha(\phi)\,F(\phi)\big(u(t_n)\big)$$

$$\ell_j^{(q-1)}(0) = \frac{1}{q}\sum_{\phi\in T_q} \alpha(\phi)\,\gamma(\phi)\,\Phi_j(\phi)\,F(\phi)\big(u(t_n)\big)$$

T_q = set of labeled trees of order q

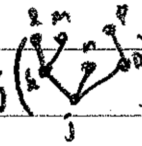
$\alpha(\phi)$ = # of ways of labeling nodes (an integer)

$\gamma(\phi)$ = product of subtree orders (another integer)

$\Phi_j(\phi)$ = "tree product" of Butcher array entries

$F(\phi)(u)$ = elementary differential

examples: $F\Big($  $\Big)(u) = D^3 f(u)\Big(D^2 f(u)\big(f(u), f(u)\big), f(u), Df(u)\big(f(u)\big)\Big)$

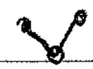
$\Phi_j\Big($  $\Big) = \sum_{k,l,m,n,o,p} a_{jk}\, a_{kl}\, a_{km}\, a_{jn}\, a_{jo}\, a_{op}$

where do elementary differentials come from?

$u' = f(u)$ 

$u'' = Df(u)u' = Df(u)f(u)$ 

$u''' = D^2 f(u)\big(f(u), f(u)\big) + Df(u)\, Df(u)\, f(u)$

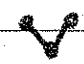
 

we've been doing this already with our PDE's

$u_t = u_{xx}$ 

$u_{tt} = u_{txx} = u_{xxxx}$ 

$u_{ttt} = u_{xxxxxx}$ 

the term corresponding to  is zero since the PDE is linear:

$f(u) = u_{xx}$

$Df(u)(v) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(u+\varepsilon v) = v_{xx}$

$D^2 f(u)(v,w) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} Df(u+\varepsilon w)(v) = \frac{d}{d\varepsilon} v_{xx} = 0$

In the formula for τ^n above, u', u'', etc. refer to the exact solution of the ODE that we got after discretizing space. To compute the truncation error of the PDE, we have to replace these with u_t, u_{tt}, etc. for the PDE. This just introduces terms in τ^n of the form

$$\frac{1}{q!}\left(\partial_t^q u_{PDE} - u_{ODE}^{(q)}\right)k^q$$

We use trees to write $\partial_t^q u_{PDE}(x_j, t_n) = \sum_{\phi \in T_q} \alpha(\phi) F_{PDE}(\phi)\left(u(x_j, t_n)\right)$

Final result: (drop n from τ^n)

$$\tau = \frac{1}{k}\left[\left(\partial_t u_{PDE} - \sum_j b_j l_j\right)k + \cdots + \left(\frac{1}{(p+1)!}\partial_t^{p+1} u_{PDE} - \frac{1}{p!}\sum_j b_j l_j^{(p)}\right)k^{p+1}\right] + O(k^{p+1})$$

$$= \frac{1}{k}\left[\tau^{(1)}k + \tau^{(2)}\frac{k^2}{2} + \cdots + \tau_n^{(p+1)}\frac{k^{p+1}}{(p+1)!}\right] + O(k^{p+1})$$

$$\tau^{(q)} = \tau^{(q)}_{space} + \tau^{(q)}_{time}$$

$$\tau^{(q)}_{space} = \sum_{\phi \in T_q} \alpha(\phi)\left[F_{PDE}(\phi)(u) - F_{ODE}(\phi)(u)\right]$$

$$\tau^{(q)}_{time} = \sum_{\phi \in T_q} \alpha(\phi) \underbrace{\left[1 - \gamma(\phi)\sum_{j=1}^{s} b_j \Phi_j(\phi)\right]}_{0 \text{ if } q \le p = \text{order of the scheme}} F_{ODE}(\phi)(u)$$

here we use the Runge-Kutta order conditions

$$\sum_{j=1}^{s} b_j \Phi_j(\phi) = \frac{1}{\gamma(\phi)} \qquad \phi \in T_q, \quad 1 \le q \le p$$

Let's bring this down to Earth by returning to the original example

PDU: $\quad u_t = u_{xx}$

ODE: $\quad u_t = Bu \quad , \qquad B = D_x^+ D_x^-$

scheme: $\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$ (trap. rule) 2nd order, so $p=2$

$$\tau = \frac{1}{k}\left[\tau^{(1)} k + \tau^{(2)} \frac{k^2}{2} + \tau^{(3)} \frac{k^3}{6} \right] + O(k^3)$$

$\underline{q=1.}$ $\quad T_1 = \{ \bullet \}$ (only one tree of order 1)

$$\tau^{(1)}_{space} = \underbrace{\alpha(\bullet)}_{1} \Big[\underbrace{F_{PDE}(\bullet)(u)}_{f(u) = u_{xx}} - \underbrace{F_{ODE}(\bullet)(u)}_{Bu} \Big]$$

$$= u_{xx} - \left[u_{xx} + \frac{h^2}{12} u_{xxxx} + \cdots \right] = -\frac{h^2}{12} u_{xxxx} + \cdots$$

$$\tau^{(1)}_{time} = 0$$

$\underline{q=2}$ $\quad T_2 = \{ \quad \}$

$$\tau^{(2)}_{space} = \underbrace{\alpha(\quad)}_{1} \Big[\overset{\text{PDE}}{\underbrace{Df(u)(f(u))}_{u_{xxxx}}} - \overset{\text{ODE}}{\underbrace{Df(u)(f(u))}_{B^2 u}} \Big]$$

$$B^2 u = B\left[u_{xx} + \frac{h^2}{12} u_{xxxx} + \cdots \right] = u_{xxxx} + \frac{h^2}{6} u_{6x} + \cdots$$

$$\tau^{(2)}_{space} = -\frac{h^2}{6} u_{6x} + \cdots \qquad \tau^{(2)}_{time} = 0$$

$\underline{q=3}$ $\quad T_3 = \{ \; , \; \}$

both operators f_{PDE} and f_{ODE} have $F(\;)(u) = 0$

$$\tau^{(3)}_{space} = \underbrace{\alpha(\;)}_{1}\Big[\underbrace{Df(u)\big(Df(u)(f(u))\big)}_{u_{6x}}^{PDE} - \underbrace{Df(u)\big(Df(u)(f(u))\big)}_{B^3 u}^{ODE}\Big]$$

$$B^3 u = B\left[u_{4x} + \frac{h^2}{6}u_{6x} + \cdots\right] = u_{6x} + \frac{h^2}{4}u_{8x} + \cdots$$

$$\tau^{(3)}_{space} = -\frac{h^2}{4}u_{8x} + \cdots$$

$$\tau^{(3)}_{time} = \underbrace{\alpha(\;)}_{1}\Big[1 - \underbrace{\gamma(\;)}_{6}\underbrace{\sum_{j=1}^{2} b_j \Phi_j(\;)}_{b^TAc}\Big]\underbrace{F_{ODE}(\;)(u)}_{B^3 u}$$

$$\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}0 & 0\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{4}$$

$$= \left(1 - \frac{3}{2}\right)B^3 u = -\frac{1}{2}\left[u_{6x} + \frac{h^2}{4}u_{8x} + \cdots\right]$$

put it all together:

$$\tau = -\frac{h^2}{12}u_{xxxx} - \frac{h^2 k}{12}u_{6x} - \frac{h^2k^2}{24}u_{8x}$$

$$\quad - \frac{k^2}{12}u_{6x} - \frac{k^2h^2}{48}u_{8x}$$

$$\tau = -\frac{h^2}{12}u_{xxxx} - \frac{k^2}{12}u_{6x} - \frac{h^2k}{12}u_{6x} - \frac{h^2k^2}{16}u_{8x} + O(h^4 + k^3)$$

this formula for τ is different than the one in the notes on page 65 because

1. in the notes we expanded about $(x_j, t_n + \frac{k}{2})$ while here we expanded about (x_j, t_n)

2. for implicit schemes, when we say "τ is what's left over when you plug the exact solution into the scheme", we really mean "initialize the scheme with the exact solution at t_n, then compare the exact solution at t_{n+1} to the result of the scheme." I did it the first way (i.e. incorrectly) in the notes.

228B, Lec 9, Spring 2011

Last time: method of lines (truncation error, Runge-Kutta order conditions)

stability

ODE after discretization in space: $u_t = Bu$

B a finite difference operator

recall linear stability analysis for Runge-Kutta:

$$y' = \lambda y, \quad \lambda \in \mathbb{C}$$

$$l_i = \lambda\left(y_n + k\sum_{j=1}^{s} a_{ij} l_j\right)$$

$$(I - k\lambda A)l = \lambda y_n e, \quad e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \; l = \begin{pmatrix} l_1 \\ \vdots \\ l_s \end{pmatrix} \in \mathbb{C}^s$$

$$y_{n+1} = y_n + kb^T l = \underbrace{\left[1 + k\lambda b^T(I - k\lambda A)^{-1} e\right]}_{R(k\lambda) = \text{stability function}} y_n$$

$$y_n = R(k\lambda)^n y_0$$

now consider one Runge-Kutta step of $u_t = Bu$

$$l_i = B\left(u^n + k\sum_{j=1}^{s} a_{ij} l_j\right)$$

we can reduce to the previous case by diagonalizing B:

$$\begin{tikzcd} u_j \arrow[r, "B = Z^{-1}GZ"] \arrow[d, "Z"] & Bu_j \\ \hat{u}(\xi) \arrow[r, "G = ZBZ^{-1}"'] & \widehat{Bu}(\xi) \arrow[u, "Z^{-1}"'] \end{tikzcd}$$

$$Zl_i = ZBZ^{-1}\left(Zu^n + k\sum_{j=1}^{s} a_{ij} Zl_j\right)$$

$$\hat{l}_i(\xi) = G(\xi)\left(\hat{u}^n(\xi) + k\sum_j a_{ij}\hat{l}_j(\xi)\right)$$

with the correspondence $y_n \leftrightarrow \hat{u}^n(\xi)$, $\lambda \leftrightarrow G(\xi)$ we obtain

$$\hat{u}^n(\xi) = R(kG(\xi))^n \hat{u}^0(\xi)$$

or

$$u_j^n = \left[R(kB)^n u^0\right]_j$$

where $R(kB) = Z^{-1} R(kG(\xi)) Z$ ← usual functional calculus trick of applying a function to a matrix or operator by diagonalizing it and applying the function to its eigenvalues.

$$A = U \Lambda U^{-1}$$
$$f(A) = U \underbrace{f(\Lambda)}_{} U^{-1}$$
$$\begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix}$$

conclusion: the method of lines is stable if $\exists\, \varepsilon, K$ s.t.

$$\|R(kB(k))^n\| \le K \quad \text{for} \quad \begin{matrix} 0 < k < \varepsilon \\ 0 \le nk \le T \end{matrix}$$

(In general, B depends on k. In that case, we write $G(\xi, k)$)

A sufficient condition:

$$|R(kG(\xi,k))| \le 1 + Ck \quad \text{for} \quad -\pi \le \xi \le \pi$$

<u>example</u>: Crank-Nicolson

<u>space</u>: $Bu_j = D_x^+ D_x^- u_j = \frac{1}{h^2}\left[u_{j+1} - 2u_j + u_{j-1}\right]$

$$G(\xi) = \frac{1}{h^2}\left[e^{i\xi} - 2 + e^{-i\xi}\right] = -\frac{2}{h^2}(1 - \cos\xi) = -\frac{4}{h^2}\sin^2\left(\frac{\xi}{2}\right)$$

<u>time</u>: $y' = \lambda y$, $y^{n+1} = y^n + \frac{k}{2}\left[\lambda y^n + \lambda y^{n+1}\right]$

$$\left(1 - \frac{k\lambda}{2}\right) y^{n+1} = \left(1 + \frac{k\lambda}{2}\right) y^n \qquad y^{n+1} = \underbrace{\frac{1 + \frac{k\lambda}{2}}{1 - \frac{k\lambda}{2}}}_{R(k\lambda)} y^n$$

<u>combined</u>: $|R(kG(\xi))| = \left|\dfrac{1 - 2\nu\sin^2(\xi/2)}{1 + 2\nu\sin^2(\xi/2)}\right| \le 1$ (unconditionally stable)

$\text{RAS} = \{ z : |R(z)| \le 1 \}$

since the trapezoidal rule is A-stable (the region of absolute stability contains the left half-plane) we will have

$$\left| R(kG(\xi)) \right| \le 1$$

provided that $\mathrm{Re}\{G(\xi)\} \le 0$. This is sometimes too much to hope for, e.g. when solving

$$u_t = u_{xx} + u$$

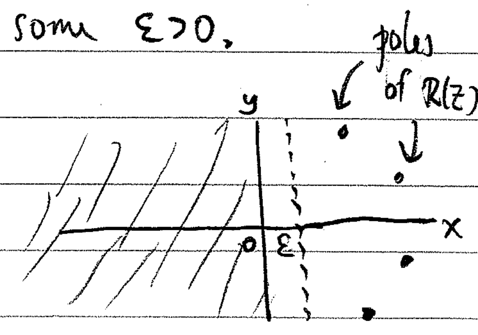
the exact solution grows, so we need $\| R(kB) \| > 1$ so the numerical solution can grow. If the Runge-Kutta method is A-stable, then $R(z) = \dfrac{P(z)}{Q(z)} = \dfrac{p_0 + p_1 z + \cdots + p_\alpha z^\alpha}{q_0 + q_1 z + \cdots + q_\beta z^\beta}$

with $\alpha \le \beta \le s$ and $|p_\beta| \le |q_\beta|$. ($p_\beta = 0$ is allowed) otherwise $|R(z)|$ would blow up $(\alpha > \beta)$ or approach $\left|\dfrac{p_\beta}{q_\beta}\right| > 1$ as $z \to \infty$. Moreover, all the poles of $R(z)$ lie in the right half plane. We conclude that

$$R'(z) = \frac{Q(z)P'(z) - P(z)Q'(z)}{Q(z)^2} = \frac{\tilde{P}}{Q^2} \qquad \deg \tilde{P} < \deg Q^2 = 2\beta$$

is bounded in $\mathbb{C}_\varepsilon^- = \{ z : \mathrm{Re}\, z \le \varepsilon \}$ for some $\varepsilon > 0$.

This leads to a simple stability condition for the method of lines.



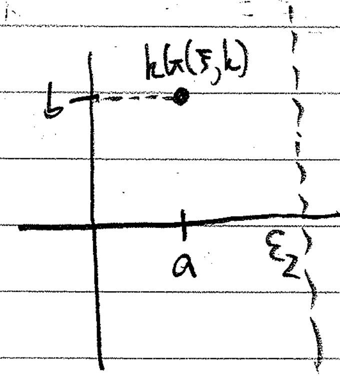
Theorem: If $\exists\, \varepsilon_1, C_1$ s.t. $\mathrm{Re}\{G(\xi,h)\} \le C_1$ for $0<k<\varepsilon_1$ and the timestepping scheme is A-stable, then the method of lines is stable.

proof: Let $\varepsilon_2 > 0$, C_2 be constants so that $|R'(z)| \le C_2$ for $z \in \mathbb{C}^-_{\varepsilon_2}$

Let $\varepsilon = \min\left(\varepsilon_1, \dfrac{\varepsilon_2}{C_1}\right)$. Write $kG(\xi,h) = a + ib$

Then for $0<k<\varepsilon$ we have

$$a = k\,\mathrm{Re}\{G(\xi,h)\} < \frac{\varepsilon_2}{C_1} C_1 = \varepsilon_2$$



$$R(a+ib) = R(ib) + \int_0^1 \underbrace{R'(ib+\theta a)\,a\, d\theta}_{\frac{d}{d\theta} R(ib+\theta a)}$$

$$|R(a+ib)| \le |R(ib)| + C_2 a \le 1 + \underbrace{C_2 C_1}_{C} k$$

$$|R(kG(\xi,h))| \le 1 + Ck. \quad \boxed{\text{Scheme is stable.}}$$

Note: we assumed $a>0$. If $a \le 0$, we know $|R(a+ib)| \le 1$ since the scheme is A-stable.

Example: $u_t = u_{xx} + u_x + u$

$$u_t = Bu, \quad Bu_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + \frac{u_{j+1} - u_{j-1}}{h} + u_j$$

$$G(\xi,k) = \frac{1}{h^2}\left[-4\sin^2\frac{\xi}{2}\right] + \frac{1}{h}\left[2i\sin\xi\right] + 1$$

still need to specify a refinement path $h(k)$

$\mathrm{Re}\{G(\xi,h)\} \le 1$, stable.

recap: $u_t = -a u_x$

each Fourier mode travels with speed a:

$$u(x,0) = e^{ix\frac{\xi}{h}} \Rightarrow u(x,t) = e^{i(x-at)\frac{\xi}{h}}$$

$$u(x,t+k) = e^{-ia\nu\xi}\, u(x,t)$$

numerical solution

$$u_j^0 = e^{ij\xi} \Rightarrow u_j^{n+1} = G(\xi)\, u_j^n$$

$$G(\xi) = \rho(\xi)\, e^{-i\alpha(\xi)\nu\xi}$$

numerical solution is exact if $\rho(\xi) = 1$, $\alpha(\xi) = a$

<u>Theorem</u>: the scheme is order r if

$$\rho(\xi) = 1 + O(\xi^{r+1}), \quad \frac{\alpha(\xi)}{a} = 1 + O(\xi^r)$$

<u>proof</u>: these conditions imply that $G(\xi) - e^{-ia\nu\xi} = O(\xi^{r+1})$

(just note that $\underset{\substack{\uparrow \\ \text{gives extra order}}}{\alpha(\xi)\nu\xi} = a\nu\xi + O(\xi^{r+1})$)

back to original def. of ξ-transform: $\hat{u}^n(\xi) = \sum_j e^{-ij\xi}$

$$\tau_j^n = \frac{1}{k}\Big[u(jh, nk+k) - \big(Bu(\cdot h, nk)\big)_j \Big]$$

$$\hat{\tau}^n(\xi) = \frac{1}{k}\underbrace{\Big[e^{-ia\nu\xi} - G(\xi) \Big]}_{A(\xi)(i\xi)^{r+1}}\, \widehat{u(\cdot h, nk)}(\xi), \qquad |A(\xi)| \le M = \text{const}$$

$$= A(\xi)\frac{h^{r+1}}{k}\Big[\Big(i\frac{\xi}{h}\Big)^{r+1} \widehat{u(\cdot h, nk)}(\xi) \Big]$$

$$h\sum_j |\tau_j^n|^2 = \frac{h}{2\pi}\int_{-\pi}^{\pi} |\hat{\tau}^n(\xi)|^2 d\xi$$

$$\overset{\circledast}{\le} (M\nu h^r)^2 \left[\frac{h}{2\pi}\int_{-\pi}^{\pi} \left| \left(i\frac{\xi}{h}\right)^{r+1} \widehat{u(\cdot h, nk)}(\xi)\right|^2 d\xi \right]$$

$$\underset{\text{to be explained}}{\Rightarrow} \quad \|\tau^n\|_{2,h} \le (M\nu h^r)\, \|u^{(r+1)}(x)\|_{L^2(\mathbb{R})} + O(h^{r+1}) \qquad \begin{matrix} \xi = h\lambda \\ d\xi = h\,d\lambda \end{matrix}$$

In the last step, we use the fact that the Z-transform is closely related to the Fourier transform:

Given $v(x)$, Let $\tilde{v}(\lambda) = \int_{-\infty}^{\infty} v(x) e^{-i\lambda x} dx$

Poisson summation formula:

$$\sum_j v(jh) e^{-ij\xi} = \sum_n \frac{1}{h} \tilde{v}\left(\frac{\xi + 2\pi n}{h}\right)$$

$$\underbrace{\hat{v}(\xi)}_{\substack{\text{Z transform} \\ \text{of sampled} \\ \text{points}}} = \underbrace{\frac{1}{h}\tilde{v}\left(\frac{\xi}{h}\right)}_{\substack{\text{fourier} \\ \text{transform}}} + \underbrace{\sum_{n\neq 0} \frac{1}{h}\tilde{v}\left(\frac{\xi + 2\pi n}{h}\right)}_{\text{aliasing error}}$$

Fourier transform of derivatives:

$$(i\lambda)^m \tilde{v}(\lambda) = \int v(x) (i\lambda)^m e^{-i\lambda x} dx \underset{\text{int by parts}}{=} \int v^{(m)}(x) e^{-i\lambda x} dx$$

$$\left(i\frac{\xi}{h}\right)^m \hat{v}(\xi) = \frac{1}{h}\widetilde{v^{(m)}}\left(\frac{\xi}{h}\right) + \sum_{n\neq 0} \frac{1}{h}\widetilde{v^{(m)}}\left(\frac{\xi + 2\pi n}{h}\right)$$

$$\text{Thus, } \left(\frac{h}{2\pi}\int_{-\pi}^{\pi}\left|\left(i\frac{\xi}{h}\right)^m \hat{v}(\xi)\right|^2 d\xi\right)^{1/2}$$

$$\le \left(\frac{h}{2\pi}\int_{-\pi}^{\pi}\left|\frac{1}{h}\widehat{v^{(m)}}\left(\frac{\xi}{h}\right)\right|^2 d\xi\right)^{1/2}$$

$$+ \left(\frac{h}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{n\neq 0}\frac{1}{h}\widehat{v^{(m)}}\left(\frac{\xi+2\pi n}{h}\right)\right|^2 d\xi\right)^{1/2}$$

2nd term bounded using $|\lambda^2 \widehat{v^{(m)}}(\lambda)| \le C_{m+2} = \int |v^{(m+2)}(x)|\,dx$

$|\widehat{v^{(m)}}(\lambda)| \le \frac{C_{m+2}}{\lambda^2} \longrightarrow$

$$\sum_{n\neq 0}\frac{1}{h}\widehat{v^{(m)}}\left(\frac{\xi+2\pi n}{h}\right) \le \frac{2}{h}\sum_{n=1}^{\infty}\frac{C_{m+2}}{\left(\frac{2\pi n-\pi}{h}\right)^2}$$

$\leftarrow |\xi+2\pi n| \ge 2\pi n - |\xi| \ge 2\pi n - \pi$

$$= \frac{2hC_{m+2}}{\pi^2}\underbrace{\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}}_{\pi^2/8} = \frac{hC_{m+2}}{4}$$

$$\therefore \text{ 2nd term} \le \frac{h^{3/2}C_{m+2}}{4}$$

1st term computed using chage of variables

$$\frac{h}{2\pi}\int_{-\pi}^{\pi}\left|\frac{1}{h}\widehat{v^{(m)}}\left(\frac{\xi}{h}\right)\right|^2 d\xi \qquad \begin{array}{l}\xi = \lambda h\\ d\xi = h\,d\lambda\end{array}$$

$$= \frac{h^2}{2\pi}\int_{-\pi/h}^{\pi/h}\frac{1}{h^2}\left|\widehat{v^{(m)}}(\lambda)\right|^2 d\lambda$$

$$\approx \frac{1}{2\pi}\int_{-\infty}^{\infty}|\widehat{v^{(m)}}(\lambda)|^2 d\lambda = \int_{-\infty}^{\infty}|v^{(m)}(x)|^2 dx$$

Plancherel's theorem

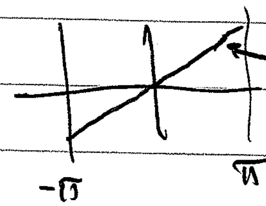
error is $O(h^3)$ since $\widehat{v^{(m)}}(\lambda) \le \frac{C_{m+2}}{\lambda^2}$

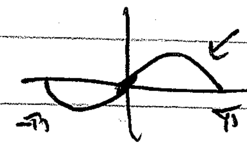
$$\text{error} \le \frac{2}{2\pi}\int_{\pi/h}^{\infty}\left(\frac{C_{m+2}}{\lambda^2}\right)^2 d\lambda = \frac{C_{m+2}^2}{\pi}\left(\frac{-\lambda^{-3}}{-3}\right)\Big|_{\pi/h}^{\infty} = \frac{C_{m+2}^2 h^3}{3\pi^4}$$

(add to discussion on Fourier collocation / pseudo-spectral methods
Page 147 of 228B notes)

<u>filtering.</u> High frequency modes are not likely to be accurate.
(not enough grid pts to resolve them).

But they get amplified the most when we take a spectral derivative
$B = D$





By contrast, D_x^0 suppresses these modes

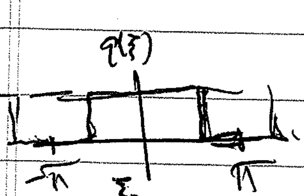
but is only 2nd order accurate $\left(ih^{-1}\xi - ih^{-1}\sin\xi = O(\xi^3) \right)$
$r+1$

A filtered version of D gives the best of both worlds:

$$G(\xi) = ih^{-1}\xi \underbrace{q(\xi)}_{\text{filter}}$$

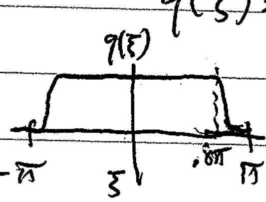
<u>Common choices:</u> $\frac{2}{3}$ rule : $q(\xi) = \begin{cases} 1 & |\xi| \le \frac{2}{3}\pi \\ 0 & \text{o.w.} \end{cases}$

36 rule: $q(\xi) = e^{-36\left(\frac{|\xi|}{\pi}\right)^{36}}$ (36th order method)



2/3 rule

avoid aliasing in $(u^2)^\wedge_l = \sum_{k=-N/2}^{N/2-1} \hat{u}_k \hat{u}_{l-k}$



36 rule

$$Du_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} ih^{-1}\xi q(\xi) e^{ij\xi}\hat{u}(\xi)\,d\xi$$

$$D^2 u_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} -h^{-2}\xi^2 q(\xi)^2 e^{ij\xi}\hat{u}(\xi)\,d\xi$$

Hyperbolic conservation laws

$$u_t + F(u)_x = 0$$

$u \in \mathbb{R}^m$, u_j = (density of) state variable (mass, momentum, energy ...)

F = flux function

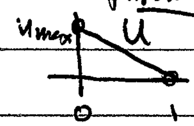
<u>examples:</u> 1. traffic flow : $\rho_t + (\rho U(\rho))_x = 0$

ρ = density of cars (# of cars per car length)

$0 \le \rho \le 1$ ($\rho = 0$: open highway, $\rho = 1$: bumper to bumper)

$U(\rho)$ = velocity $\left(\begin{array}{l} U(0) = \text{speed limit} \\ U(1) = 0 \text{ mph} \end{array} \right)$

popular model:



$F(\rho) = \rho U(\rho)$ = flux of cars (# cars/sec passing a point)

2. compressible gas dynamics

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho v \\ E \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{bmatrix} = 0$$

ρ = density

v = velocity

ρv = momentum

E = energy

p = pressure e.g. $p(\rho) = K\rho^\gamma$

equation of state: (ideal gas law)

3. Isentropic Euler (drop energy)

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho v \end{pmatrix} + \begin{pmatrix} \rho v \\ \rho v^2 + K\rho^\gamma \end{pmatrix}_x = 0$$

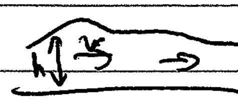
4. Shallow water eqs.

$$h_t + (vh)_x = 0$$
$$v_t + \left(\tfrac{v^2}{2} + gh\right)_x = 0$$

v = average horizontal velocity

h = height

g = gravity



↑ this is the nonphysical version. better:

$$\begin{pmatrix} h \\ hu \end{pmatrix}_t + \begin{pmatrix} uh \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}_x = 0$$

5. Burgers' equation

$$u_t + uu_x = 0 \qquad F(u) = \tfrac{1}{2}u^2 \qquad F(u)_x = uu_x$$

(inviscid limit of viscous Burgers: $u_t + uu_x = \varepsilon u_{xx}$

itself a 1d toy model of Navier-Stokes

$$\rho(u_t + u\cdot\nabla u) = -\nabla p + \mu\Delta u$$
$$\nabla\cdot u = 0 \quad)$$

---

continuity equation: $\rho_t + (\rho v)_x = 0$

$a(t)$ —— $b(t)$ $\qquad \dot a = v(a), \quad \dot b = v(b)$ $\qquad$ follow the material

$$0 = \frac{d}{dt}\int_a^b \rho\, dx = \int_a^b \rho_t\, dx + \rho(b)\underbrace{\dot b}_{u(b)} - \rho(a)\underbrace{\dot a}_{u(a)}$$

$$= \int_a^b \rho_t + (\rho v)_x\, dx$$

similarly,

$$p(a) - p(b) = \frac{d}{dt}\int_a^b \rho v\, dx \qquad \text{(momentum equation)}$$

$$\Longrightarrow \int_a^b \rho_t + (\rho v^2 - p)_x\, dx$$

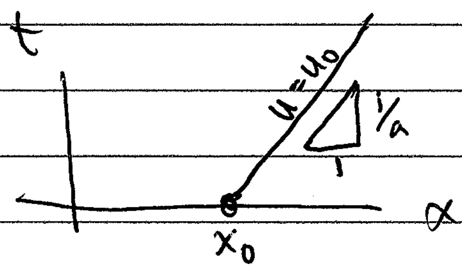
Characteristics.

$$u_t + F(u)_x = 0 \;\rightarrow\; u_t + \underbrace{F'(u)}_{a(u)} u_x = 0$$

Let $x(t)$ evolve in time and note that

$$\frac{d}{dt} u(x(t),t) = u_x \dot{x} + u_t = 0 \qquad \text{(u remains constant)}$$

provided $\dot{x}(t) = a\big(u(x(t),t)\big) = a(u_0)$



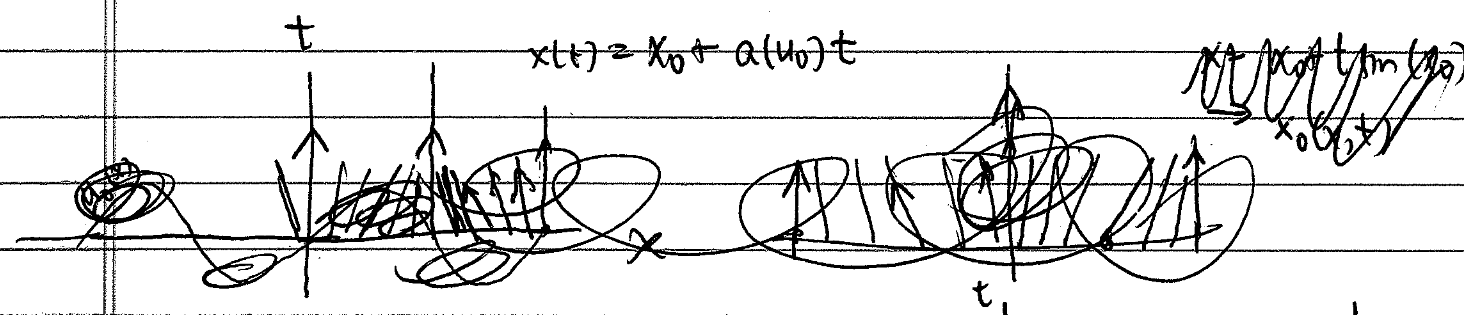
$$\frac{dt}{dx} = \left(\frac{dx}{dt}\right)^{-1} = \frac{1}{a}$$

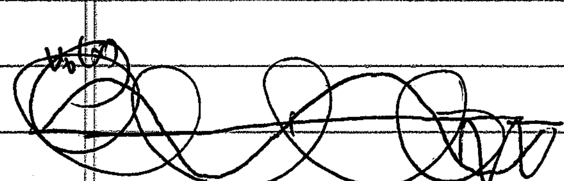
example: Burgers' eqn.

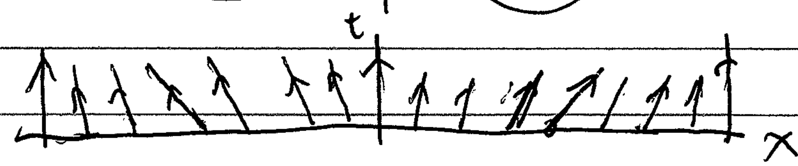
$u_t + u u_x = 0 \qquad a = u$

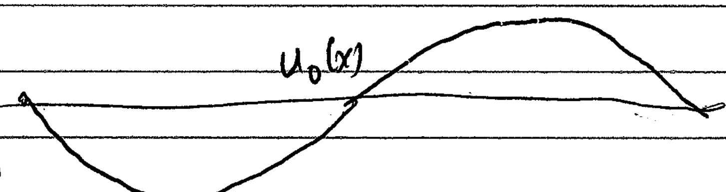
$u(x,0) = u_0(x) = \sin x$

$x(t) = x_0 + a(u_0)t$









$u(x_0 + t\sin(x_0), t) = \sin x_0$

$x = x_0 + t\sin(x_0) \rightarrow x_0(x,t)$ solve implicitly

$\longrightarrow$ $u(x,t) = \sin x_0(x,t)$

exact solution (valid until characteristics cross, after which you can't solve for $x_0(x,t)$ uniquely) general case: $u_t + G(u_x, u, x, t) = 0 \quad (\circledast)$, $G(p, z, x, t)$ given.

solution propagates along characteristics
$$\left\{ \begin{aligned} &x(t) \\ &p(t) = u_x(x(t), t) \\ &q(t) = u_t(x(t), t) \\ &z(t) = u(x(t), t) \end{aligned} \right\}$$
satisfying the ODEs:

$$\left[\begin{aligned} \dot{x} &= G_p(p, z, x, t) \\ \dot{p} &= -G_x - G_z p \\ \dot{q} &= -G_t - G_z q \\ \dot{z} &= p G_p + q \end{aligned} \right.$$

derivation

$$\dot{p} = u_{xx} \dot{x} + u_{xt}$$

$$\dot{q} = u_{tx} \dot{x} + u_{tt}$$

use $\circledast$ to eliminate 2nd derivatives:

$$\underbrace{u_{tx} + G_p u_{xx}}_{\dot{p}} + G_z u_x + G_x = 0$$

define $\dot{x} = G_p$

$$\underbrace{u_{tt} + G_p u_{xt}}_{\dot{q}} + G_z u_t + G_t = 0$$

finally, $\dot{z} = u_x \dot{x} + u_t = p \dot{x} + q$

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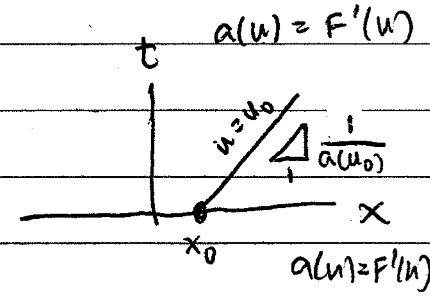
no office hours next week.

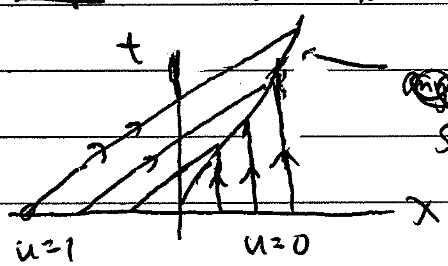
last time: $u_t + F(u)_x = 0$



solutions are constant along characteristics

what should you do when characteristics cross?

<u>example:</u> $u_t + uu_x = 0$, $u(x,0) = \begin{cases} 1 & x<0 \\ 0 & x>0 \end{cases}$



$s(t)$
shock =

~~to the left of the shock, $u(x,t)=1$~~
--- right ------ $u(x,t)=0$

the PDE is satisfied in each region, but the solution is discontinuous across the shock.

to make sense of discontinuous solutions, we need a more general definition:

for any fixed interval $a<x<b$

weak solutions, version 1:

$$\frac{\partial}{\partial t}\int_a^b u(x,t)\,dx = F(u(a,t)) - F(u(b,t))$$

this holds for smooth solutions since $\int_a^b u_t\,dx = \int_a^b -F(u)_x\,dx = -F(u)\Big|_{x=a}^{x=b}$

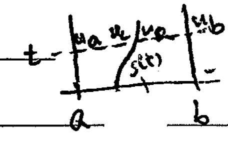
but it also makes sense if there are shocks inside $[a,b]$

and is physically relevant!

!!!

suppose $[a,b]$ contains only one shock $s(t)$ for $t_0<t<t_1$

<u>Shock speed:</u> Let $u_L = u(s(t)^-, t)$, $u_R = u(s(t)^+, t)$,



$u_a = u(a,t)$, $u_b = u(b,t)$

then

$$\frac{\partial}{\partial t}\int_a^b u(x,t)\,dx = \frac{\partial}{\partial t}\left[\int_a^s u\,dx + \int_s^b u\,dx\right]$$

$$\overset{\text{Leibniz}}{=} F(u_a) - F(u_L) + u_L\dot{s}$$

$$+ F(u_R) - F(u_b) - u_R\dot{s} \overset{\text{weak sol'n}}{=} F(u_a) - F(u_b)$$

$$\therefore \quad \dot{s} = \frac{F(u_R) - F(u_L)}{u_R - u_L} \qquad \text{Rankine-Hugoniot condition}$$

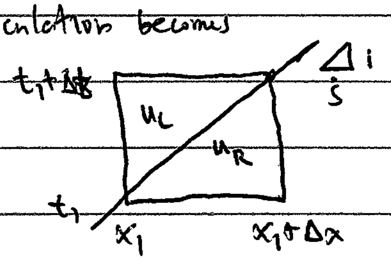
$$\dot{s} = \frac{[\![F(u)]\!]}{[\![u]\!]}$$

(for systems, can't divide by $[\![u]\!]$ since it's a vector. need $(u_R-u_L)\dot{s} = F(u_R)-F(u_L)$ in that case)

weak solution, version 2: over any rectangle $x_1 < x < x_2$, $t_1 < t < t_2$

$$\int_{x_1}^{x_2} u(x,t_2) - u(x,t_1)\,dx + \int_{t_1}^{t_2} F(u(x_2,t)) - F(u(x_1,t))\,dt = 0$$

shock speed calculation becomes



$$(u_L - u_R)\Delta x + (F(u_R) - F(u_L))\Delta t = 0$$

$$\dot{s} = \frac{\Delta x}{\Delta t} =$$

weak sol'n, version 3: consider a test function $\phi(x,t)$ with cpt spt.

skip
$$\int_0^\infty u_t\phi\,dt = u\phi\Big|_0^\infty - \int_0^\infty u\phi_t\,dt = -u\phi\Big|_{t=0} - \int_0^\infty u\phi_t\,dt$$

$$\int_0^\infty\int_{-\infty}^\infty [u_t + F(u)_x]\,\phi(x,t)\,dx\,dt = 0$$

$$\Rightarrow \int_0^\infty\int_{-\infty}^\infty u\phi_t + F(u)\phi_x\,dx\,dt + \int_{-\infty}^\infty u(x,0)\,\phi(x,0)\,dx = 0$$

$$\forall \phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$$

version 2 is obtained by setting $\phi \simeq \chi_{[x_1,x_2]}\chi_{[t_1,t_2]}$

warning: apparently equivalent equations can have different shock speeds

$u_t + uu_x = 0 \rightarrow$

(1) $u_t + \left(\frac{u^2}{2}\right)_x = 0$

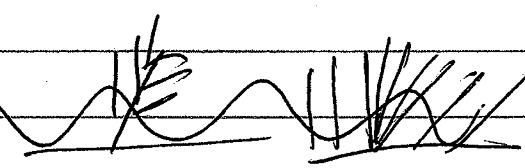
(2) $uu_t + u^2 u_x = 0 \rightarrow \left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3}\right)_x = 0$

(1) $\dot{s} = \frac{F(u_R) - F(u_L)}{u_R - u_L} = \frac{[\![\frac{1}{2}u^2]\!]}{[\![u]\!]} = \frac{1}{2}(u_R + u_L)$

(2) $\dot{s} = \frac{[\![\frac{1}{3}u^3]\!]}{[\![\frac{1}{2}u^2]\!]} = \frac{2}{3}\frac{u_R^3 - u_L^3}{u_R^2 - u_L^2} = \frac{2}{3}\frac{u_R^2 + u_R u_L + u_L^2}{u_R + u_L}$

$$\dot{s}_2 - \dot{s}_1 = \frac{1}{6}\frac{(u_R - u_L)^2}{u_R + u_L} \neq 0 \quad \text{when } u_L \neq u_R$$

next time: entropy condition



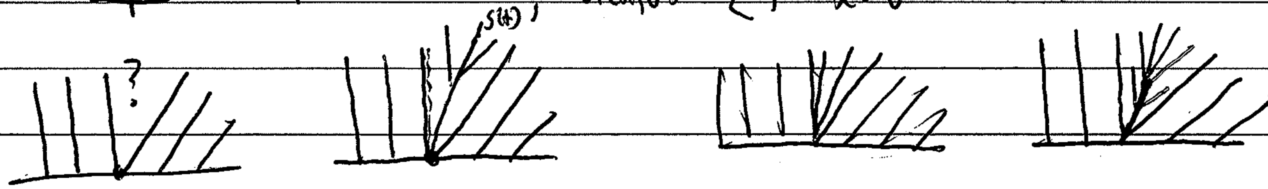
(1) is considered physically relevant since we consider u rather than u^2 to be a density

(2) can be used as an entropy function (more later)

rarefaction waves.

characteristics cross when they move toward each other, but they can also diverge, leaving a gap where the solution is undetermined.

example: $u_t + uu_x = 0$, $u(x,0) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}$



option 1 — insert a shock

option 2 — rarefaction wave

option 3 — mixture of both

option 1 yields a perfectly valid weak solution provided $s(t)$ satisfies the Rankine-Hugoniot condition.

option 2 is a similarity solution: $x \in \mathbb{R}, t > 0$ (u is constant along rays $x = ct$)

$$u(x,t) = \phi(x/t), \quad \cancel{\phi(x) = \begin{cases} 0 & x<0 \\ x & 0 \le x \le 1 \\ 1 & 1 \le x \end{cases}}$$

~~to figure out ϕ, set $t = h$,~~ following characteristics (from the initial condition) we see that $\phi(x) = \begin{cases} \cancel{\phi(x) = 0} & \cancel{x \le 0} \\ \cancel{\phi(x) = 1} & \cancel{x \ge 1} \end{cases}$

to figure out ϕ, ~~for $0 \le x \le 1$~~, plug into PDE: $u_t + uu_x = 0$

$$\frac{\partial}{\partial t}\phi\left(\frac{x}{t}\right) + \phi\left(\frac{x}{t}\right)\frac{\partial}{\partial x}\phi\left(\frac{x}{t}\right) = 0$$

$$\cancel{-\frac{x}{t^2}\phi' + \phi \cdot \frac{1}{t}\phi' = 0 \quad \frac{\phi}{t} = \frac{x}{t^2} \quad \phi\left(\frac{x}{t}\right) = \frac{x}{t}}$$

$$\phi'\left(\frac{x}{t}\right)\left[-\frac{x}{t^2} + \frac{1}{t}\phi\left(\frac{x}{t}\right)\right] = 0 \rightarrow \begin{array}{l} \phi'\left(\frac{x}{t}\right) = 0 \\ \text{or } \phi\left(\frac{x}{t}\right) = \frac{x}{t} \end{array}$$

to figure out ϕ, plug into PDE

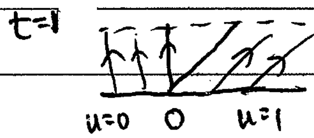
$$\cancel{u_t + u u_x = 0}$$

$$\frac{\partial}{\partial t}\phi\left(\frac{x}{t}\right) + \phi\left(\frac{x}{t}\right)\frac{\partial}{\partial x}\phi\left(\frac{x}{t}\right) = 0$$

$$\phi'\left(\frac{x}{t}\right)\left[-\frac{x}{t^2} + \frac{1}{t}\phi\left(\frac{x}{t}\right)\right] = 0$$

either $\phi'\left(\frac{x}{t}\right) = 0$ or $\phi\left(\frac{x}{t}\right) = \frac{x}{t}$

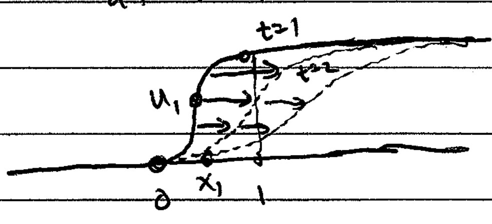
following characteristics (from the initial condition) to $t=1$ determines which case to use. ~~in our case:~~



$$\phi(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & 1 \le x \end{cases}$$

$=$

In option 3, we could imagine connecting $u=0$ on the left to $u=1$ on the right using any other continuous function at $t=1$



$$u(x,1) = \phi(x) = \begin{cases} 0 & x \le 0 \\ \text{arbitrary} & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

$t=1$.

at any later time, the solution remains smooth.

~~$u(x_1,1) = u_1$~~
~~$\Rightarrow u(x_1 + u_1(t-1), t) = u_1$~~

But if you go backward, a shock will form before you reach $t=0$.

following characteristics from $t=1$, we find that

$$u(x+\phi(x)(t-1),\, t) = \phi(x)$$

so

$$u_x(x+\phi(x)(t-1),\, t)\ \underbrace{[1+\phi'(x)(t-1)]}_{\circledast} = \phi'(x)$$

If $\exists x^*$ s.t. $\phi'(x^*) > 1$, then $\circledast = 0$ before t reaches 0 (say at t^*)
and $u_x(\ldots, t^*) = \infty$

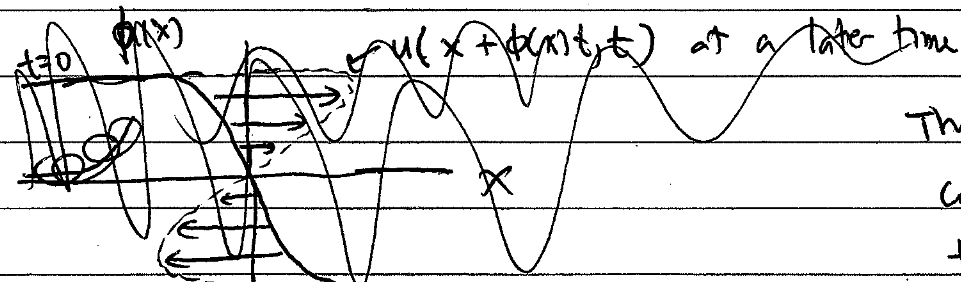
By the mean value theorem, $\circledast$ if $\phi(x) > x$ for some $x \in (0,1)$, then $\exists\ x^* \in (0,x)$ s.t. $\phi'(x^*) = \dfrac{\phi(x)-0}{x-0} > 1$

so the only solution without shocks for $0<t<1$ is the similarity solution (rarefaction wave).

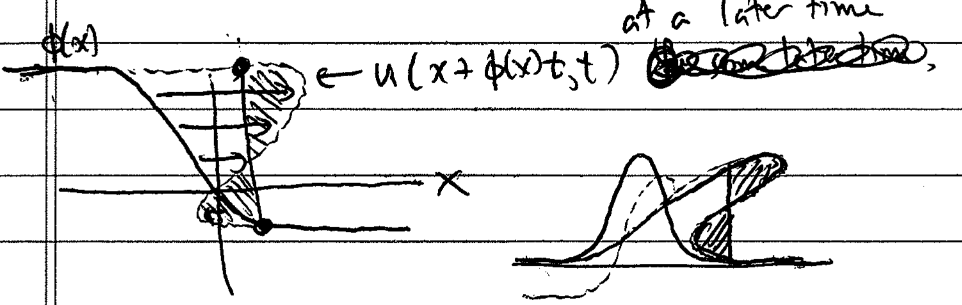
$\Rightarrow$

equal area rule:

formulas like $u(x+\phi(x)t,\, t) = \phi(x)$
make sense if you allow the solution to be multi-valued.



The Rankine-Hugoniot conditions replaces the overturning wave with a discontinuity that cuts off the same area to the right and left



entropy condition: want a condition to make weak solutions unique. Should be ① equivalent to the vanishing viscosity solution ② easy to check.

__Idea:__ require that characteristics __enter__ shocks (rather than emanate from them)

__Lax entropy condition:__ (for scalar, __convex__ conservation laws) $f''(x)>0$ for all x. (concave also OK)

every discontinuity $s(t)$ of the solution satisfies

$$f'(u_L) > s'(t) > f'(u_R)$$

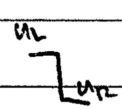
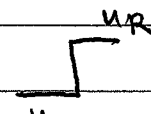
where $u_L = u(s(t)^-, t)$, $u_R = u(s(t)^+, t)$

in words: characteristics to the left travel faster than the shock, which travels faster the characteristics to the right.

This rules out everything but the rarefaction fan when the initial condition has a discontinuity with $f'(u_L) < f'(u_R)$ (there must be no shock in that case)

Oleinik entropy condition: u is the entropy solution (assuming $f''(u)>0$) if $\exists\, E>0$ s.t. $\forall\, a>0$, $t>0$ and $x\in\mathbb{R}$,

$$\frac{u(x+a,t)-u(x,t)}{a} < \frac{E}{t}$$

this allows discontinuities  but not  at positive times.

(note $f''(u)>0$ means increasing u increases $f'(u)$)

entropy functions

$\eta(u)$ = entropy function

$\psi(u)$ = " flux

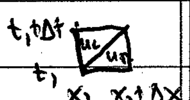
when u is smooth, $\eta(u)_t + \psi(u)_x = 0$

(entropy is conserved)

at an (admissible / inadmissible) shock, entropy (decreases / increases) (sign convention opposite of physical entropy.)

$$\boxed{\text{entropy condition: } \eta(q)_t + \psi(q)_x \le 0, \text{ integrated over a rectangle}}$$

$$\int_{x_1}^{x_2} \eta(u(x,t_2))\,dx \le \int_{x_1}^{x_2} \eta(u(x,t_1))\,dx + \int_{t_1}^{t_2} \psi(q(x_1,t))\,dt - \int_{t_1}^{t_2} \psi(q(x_2,t))\,dt$$



$$\Delta x(\eta(u_L) - \eta(u_R)) + \Delta t(\psi(u_R) - \psi(u_L)) \le 0$$

$$s'(x)(\eta(u_R) - \eta(u_L)) \ge \psi(u_R) - \psi(u_L)$$

$$s'(x)[\![\eta(u)]\!] \ge [\![\psi]\!]$$

example: Burgers' eq. $\eta(u) = u^2$

$\psi(u) = \frac{2}{3}u^3$

$$s'(x) = \frac{u_L + u_R}{2}, \quad [\![\eta]\!] = u_R^2 - u_L^2, \quad [\![\psi]\!] = \frac{2}{3}(u_R^3 - u_L^3)$$

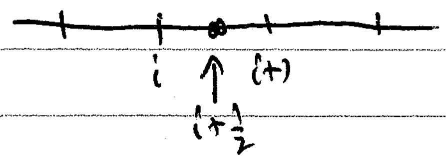
requirement: $\frac{1}{2}(u_L + u_R)(u_R^2 - u_L^2) \ge \frac{2}{3}(u_R^3 - u_L^3)$

$$\frac{1}{6}(u_L - u_R)^3 \ge 0$$

need $u_L > u_R$ to satisfy entropy condition.

conservation form: $\left(\text{equation: } u_t + F(u)_x = 0\right)$

$$\frac{1}{k}\left(u_i^{n+1} - u_i^n\right) = -\frac{1}{h}\left[G^n_{i+\frac{1}{2}} - G^n_{i-\frac{1}{2}}\right]$$



q = # of grid pts on each side to use in numerical flux function G

$$\left.\begin{aligned} G^n_{i+\frac{1}{2}} &= G(u^n_{i-q+1}, \dots, u^n_{i+q}) \\ G^n_{i-\frac{1}{2}} &= G(u^n_{i-q}, \dots, u^n_{i+q-1}) \end{aligned}\right\}$$ $\boxed{\text{requirement: } G(u,u,\dots,u) = F(u)}$

examples:

$\frac{k}{h} = \nu$

Lax-Friedrichs (LxF): $\quad u_i^{n+1} = \frac{1}{2}\left(u^n_{i-1} + u^n_{i+1}\right) - \frac{k}{2h}\left[F^n_{i+1} - F^n_{i-1}\right]$

$$G(u_0, u_1) = -\frac{1}{2\nu}(u_1 - u_0) + \frac{1}{2}\left[F(u_1) + F(u_0)\right]$$

check: $G(u,u) = F(u)$ ✓

$$-\frac{1}{h}\left[G^n_{i+\frac{1}{2}} - G^n_{i-\frac{1}{2}}\right] = \frac{1}{2k}\left[(u^n_{i+1} - u^n_i) - (u^n_i - u^n_{i-1})\right]$$
$$- \frac{1}{2h}\left[(F_{i+1} + F_i) - (F_i - F_{i-1})\right]$$

$$= \frac{1}{k}\left[\frac{u^n_{i+1} + u^n_{i-1}}{2} - \frac{k}{2h}\left(F^n_{i+1} - F^n_{i-1}\right) - u^n_i\right]$$

$$= \frac{1}{k}\left[u_i^{n+1} - u_i^n\right]$$ ✓

Lax-Wendroff

$$u(x,t+k) = u + k u_t + \frac{k^2}{2} u_{tt} + \cdots$$

$$u_t = -F(u)_x$$

$$u_{tt} = -\left[F'(u) u_t\right]_x = \left[\underbrace{F'(u)}_{a(u)} F(u)_x\right]_x$$

$$F(u)_x \approx D_0[F(u)]$$

$$\left[a(u) F(u)_x\right]_x \approx D^+\left[a_{i-\frac{1}{2}} D^- F(u)\right]$$

$$= \frac{1}{h^2}\left[a_{i+\frac{1}{2}}(F_{i+1} - F_i) - a_{i-\frac{1}{2}}(F_i - F_{i-1})\right]$$

$$a_{i+\frac{1}{2}} = \frac{1}{2}(a_i + a_{i+1})$$

<u>scheme:</u>

$$u_i^{n+1} = u_i^n - \frac{\nu}{2}\left[F_{i+1} - F_{i-1}\right] + \frac{\nu^2}{2}\left[a_{i+\frac{1}{2}}(F_{i+1} - F_i) - a_{i-\frac{1}{2}}(F_i - F_{i-1})\right]$$

numerical flux:

$$G_{i+\frac{1}{2}} = \frac{1}{2}(F_i + F_{i+1}) - \frac{\nu}{2} a_{i+\frac{1}{2}}(F_{i+1} - F_i)$$

<u>check:</u> $G(u,u) = \frac{1}{2}(F + F) - \frac{\nu}{2} a(F - F) = F(u)$ ✓

$$\frac{k}{h}\left(G_{i+\frac{1}{2}} - G_{i-\frac{1}{2}}\right) = \frac{\nu}{2}(F_{i+1} - F_{i-1}) - \frac{\nu^2}{2}\left[a_{i+\frac{1}{2}}(F_{i+1} - F_i) - a_{i-\frac{1}{2}}(F_i - F_{i-1})\right]$$

$$= -\left(u_i^{n+1} - u_i^n\right)$$ ✓

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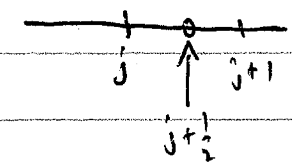
last time: • 3 ways to formulate entropy condition

• schemes in conservation form

equation: $u_t + F(u)_x = 0$

scheme: $u_j^{n+1} = u_j^n - \frac{k}{h}\left[G^n_{j+\frac{1}{2}} - G^n_{j-\frac{1}{2}} \right]$



$$G^n_{j+\frac{1}{2}} = \underbrace{G}_{\text{numerical flux function}}(u_j^n, u_{j+1}^n)$$

$G(u,u) = F(u)$ (consistency)

examples:

Lax-Friedrichs: (LxF)

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) - \frac{\nu}{2}\left[F_{j+1}^n - F_{j-1}^n \right]$$

$$G(u_j, u_{j+1}) = \underbrace{\frac{1}{2}\left[F(u_{j+1}) + F(u_j) \right]}_{\text{flux function of centered scheme}} - \underbrace{\frac{1}{2\nu}(u_{j+1} - u_j)}_{\text{adds dissipation to the centered scheme, making it stable}}$$

Lax-Wendroff (LW)

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}\left[F_{j+1} - F_{j-1} \right] + \underbrace{\frac{\nu^2}{2}\left[a_{j+\frac{1}{2}}(F_{j+1} - F_j) - a_{j-\frac{1}{2}}(F_j - F_{j-1}) \right]}_{\text{approximation of next term in Taylor expansion}}$$

$$G = \frac{1}{2}(F_{j+1} + F_j) - \frac{\nu}{2} a_{j+\frac{1}{2}}(F_{j+1} - F_j)$$

$a_{j+\frac{1}{2}} = \frac{1}{2}(a_j + a_{j+1})$

$a(u) = f'(u)$

LxF smoothes shocks too much

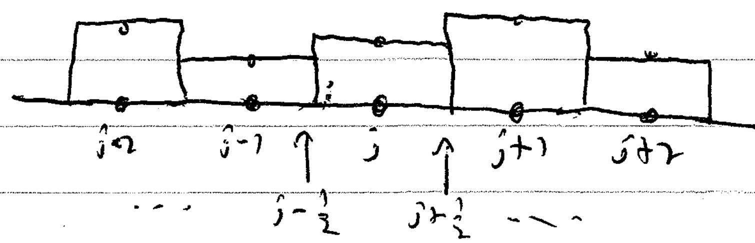
LW has too many oscillations

Breakthrough idea: Godunov's method

REA algorithm: reconstruct - evolve - average

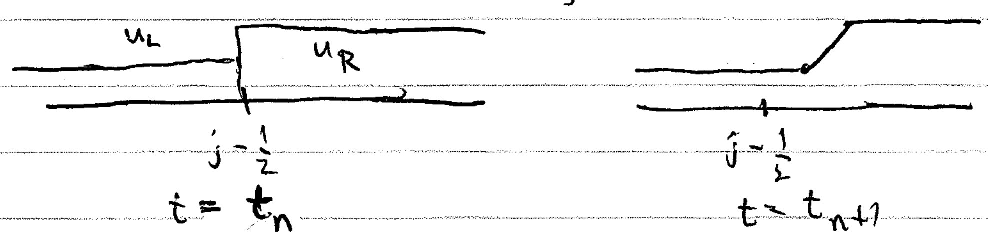
<u>reconstruct</u>: build a piecewise polynomial defined for all x from cell averages

Godunov used a piecewise constant reconstruction:



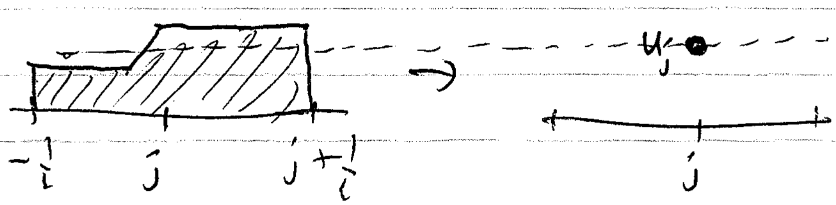
<u>evolve</u>: solve the reconstructed problem exactly from t_n to t_{n+1}

(for piecewise constant reconstruction, this involves solving a Riemann problem at each cell boundary

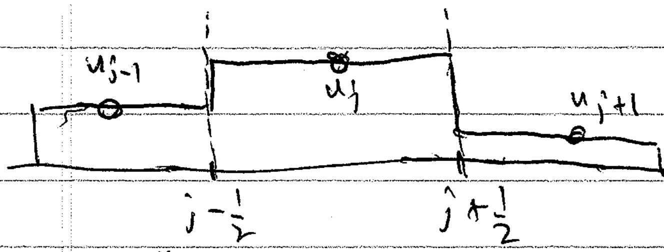


CFL prevents Riemann problem at $j-\frac{1}{2}$ from affecting $u_{j-\frac{3}{2}}$ and $u_{j+\frac{1}{2}}$

<u>average</u>: replace u_j with new cell average



usually easier to evolve and average via weak form of PDE:



$$\underbrace{\frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t_{n+1})\, dx}_{u_j^{n+1}} = \underbrace{\frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t_n)\, dx}_{u_j^n} + \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} \left[F(u(x_{j-\frac{1}{2}}, t)) - F(u(x_{j+\frac{1}{2}}, t))\right] dt$$

the Riemann problems at $x_{j-\frac{1}{2}}$ and $x_{j+\frac{1}{2}}$ are similarity solutions $\left(u = \phi\left(\frac{x - x_{j-\frac{1}{2}}}{t - t_n}\right)\right)$, so they are constant along the rays $x = x_{j-\frac{1}{2}}$ and $x = x_{j+\frac{1}{2}}$.

results

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left[F(\underbrace{u^{\downarrow}_{j-1/2}}) - F(u^{\downarrow}_{j+\frac{1}{2}})\right]$$

(Leveque's notation for solution of Riemann problem along ray $x = x_{j-\frac{1}{2}}$)

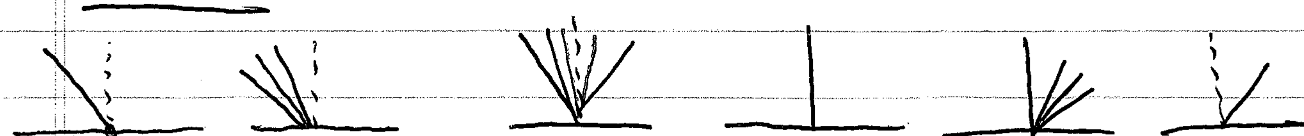
automatically in conservation form

$$G(u_j, u_{j+1}) = F(u^{\downarrow}(\underbrace{u_j, u_{j+1}}_{\text{left and right states in Riemann problem.}}))$$

usually $u^{\Downarrow}(u_L, u_R)$ is just u_L or u_R. (scheme reverts to upwind)

If F is convex ($F'' > 0$), the only exception is the transonic rarefaction, $u_L < u_s < u_R$, where u_s satisfies $a(u_s) = 0$.

<u>six cases:</u>



shock moving left: $u^{\Downarrow} = u_R$

rarefaction moving left: $u^{\Downarrow} = u_R$

transonic rarefaction: $u^{\Downarrow} = u_s$

stationary shock: $u^{\Downarrow}$ is ambiguous but $F(u_L) = F(u_R)$ so $F(u^{\Downarrow})$ is not ambiguous

rarefaction moving right: $u^{\Downarrow} = u_L$

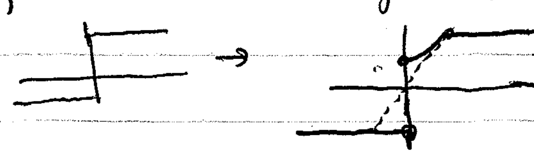
shock moving right: $u^{\Downarrow} = u_L$

in all cases,

$$F^n_{j-\frac{1}{2}} = \begin{cases} \min\limits_{u_{j-1} \leq u \leq u_j} F(u) & \text{if } u_{j-1} \leq u_j \\ \max\limits_{u_{j-1} \geq u \geq u_j} F(u) & \text{if } u_{j-1} \geq u_j \end{cases}$$

this formula is also valid if f is concave ($F''(x) < 0$) or even in the nonconvex case where there could be several stagnation points.

<u>note</u>: it's very important to treat the transonic rarefaction correctly. Otherwise, Godunov can converge to the wrong weak solution.



Glimm's method (random choice method)

- set up a piecewise constant Riemann problem $u(x) = u_j$, $x_{j-\frac{1}{2}} \leq x \leq x_{j+\frac{1}{2}}$
- evolve exactly from t_n to t_{n+1}.
- instead of defining u_j^{n+1} to be the average of the exact solution over the cell, we now choose a random point in $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ and use the value of the exact solution at that point for u_j^{n+1}.

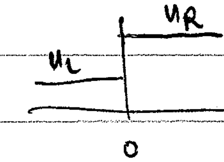
Last time:

Godunov: $u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(G_{j-\frac{1}{2}} - G_{j+\frac{1}{2}}\right)$

$$G_{j+\frac{1}{2}} = G(u_j, u_{j+1})$$

$$G(u_L, u_R) = F\left(u^{\vee}(u_L,u_R)\right) = \begin{cases} \min\limits_{u_L \le u \le u_R} F(u) & u_L \le u_R \\ \max\limits_{u_L \ge u \ge u_R} F(u) & u_L \ge u_R \end{cases}$$

$u^{\vee}(u_L,u_R)$ = exact solution of Riemann problem at $x=0,\ t>0$

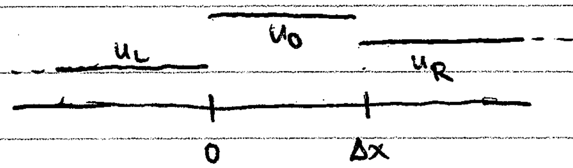


(recall exact solution is of the form $u(x,t) = \phi(x/t)$)

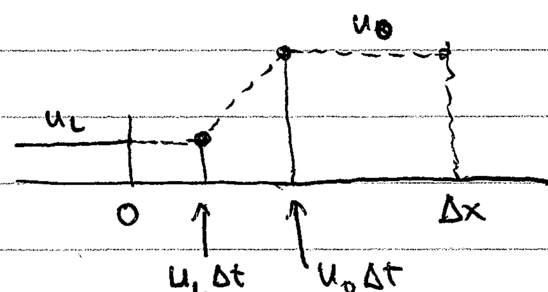
Glimm: $u_j^{n+1} = U(u_{j-1}, u_j, u_{j+1}, \Delta x\ ;\ \theta_j \Delta x, \Delta t)$

θ_j: random point $\theta_j \in [0,1]$

$U(u_L, u_0, u_R, \Delta x\ ;\ x, t)$ = exact solution with initial conditions of the form



example: Burgers'
$0 < u_L < u_0$
$0 < \frac{u_0+u_R}{2}$



$$U(x,t) = \begin{cases} u_L & 0 \le x \le u_L \Delta t \\ \dfrac{u_0\Delta t - x}{(u_0-u_L)\Delta t} u_L + \dfrac{x - u_L\Delta t}{(u_0-u_L)\Delta t} u_0 & u_L\Delta t \le x \le u_0 \Delta t \\ u_0 & u_0 \Delta t \le x \le \Delta x \end{cases}$$

Cases: $u_L = u_0$

$u_L > u_0 \begin{Bmatrix} s>0 \\ s \le 0 \end{Bmatrix}$ $s = \frac{u_L+u_0}{2}$

$u_0 = u_R$

$u_0 > u_R \begin{Bmatrix} s \ge 0 \\ s<0 \end{Bmatrix}$ $s = \frac{u_0+u_R}{2}$

$u_L < u_0 \begin{Bmatrix} 0 \le u_L \\ u_L \le 0 \le u_0 \\ u_0 \le 0 \end{Bmatrix}$

$u_0 < u_R \begin{Bmatrix} 0 \le u_0 \\ u_0 < 0 \le u_R \\ u_R < 0 \end{Bmatrix}$

Convergence

monotone schemes $\quad u_j^{n+1} = H(u_{j-q}^n, \dots, u_{j+q}^n)$

requirement: H is an increasing function of each of its arguments (separately)

equivalently: $\dfrac{\partial u_i^{n+1}}{\partial u_j^n} \geq 0 \quad \forall\, i,j \in \mathbb{Z}$

example: LxF $\quad u_j^{n+1} = \frac{1}{2}[u_{j+1} + u_{j-1}] - \frac{\nu}{2}[\underbrace{F_{j+1}}_{F(u_{j+1})} - F_{j-1}]$

$$\frac{\partial u_j^{n+1}}{\partial u_{j+1}^n} = \frac{1}{2} - \frac{\nu}{2} a(u_{j+1}^n) \geq 0 \quad \text{if} \quad \nu \leq \frac{1}{|a|} \quad (\text{CFL})$$

$$\frac{\partial u_j^{n+1}}{\partial u_{j-1}^n} = \frac{1}{2} + \frac{\nu}{2} a(u_{j-1}^n) \geq 0 \quad \text{if} \quad \nu \leq \frac{1}{|a|} \quad (\text{CFL})$$

$$\frac{\partial u_j^{n+1}}{\partial u_i^n} = 0 \quad \text{if } i \notin \{j-1, j+1\}$$

Godunov's method is also monotone (can check all cases using $u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(F(u^{\vee}_{j-\frac{1}{2}}) - F(u^{\vee}_{j+\frac{1}{2}})\right)$)

theorem: If a finite difference scheme in conservation form is monotone along a refinement path (typically $\frac{k}{h} = \nu$) then the numerical solution converges to the vanishing viscosity solution (i.e. satisfies the entropy condition) as $k \to 0$.

Heuristic proof in special case of LxF: $\nu = \frac{k}{h}$

$$u^{n+1} - \frac{1}{2}\left(u_{j+1} + u_{j-1}\right) + \frac{\nu}{2}\left(F_{j+1} - F_{j-1}\right) = 0$$

Q: what PDE are we really solving here?

Taylor expand $u(x,t)$

$$u + ku_t + \frac{k^2}{2}u_{tt} - \frac{1}{2}\begin{pmatrix} u + hu_x + \frac{h^2}{2}u_{xx} + \cdots \\ + u - hu_x + \frac{h^2}{2}u_{xx} + \cdots \end{pmatrix} + \frac{\nu}{2}\left(2h\, F(u)_x + \cdots\right) = 0$$

$$k\left(u_t + F(u)_x\right) + \frac{k^2}{2}\left(u_{tt} - \frac{1}{\nu^2}u_{xx}\right) + O(k^3) = 0$$

$$u_t + F(u)_x = O(k)$$

$$u_{tt} = -F(u)_{tx} + O(k) = -\left(F'(u)u_t\right)_x + O(k) \quad \text{another } O(k) \text{ error}$$

$$= \left(F'(u)\underbrace{F(u)_x}_{F'(u)u_x}\right)_x + O(k)$$

$$= \left(a(u)^2 u_x\right)_x + O(k)$$

so the numerical solution satisfies

$$u_t + F(u)_x = \frac{k}{2}\left[\underbrace{\left(\frac{1}{\nu^2} - a(u)^2\right)}_{\text{positive when CFL condition satisfied}} u_x\right]_x + O(k^2)$$

RHS is like a viscous term εu_{xx}, but with a spatially varying diffusion constant.

expect to obtain the vanishing viscosity solution as $k \to 0$.

---

unfortunately, monotone schemes are at most first order.

systems. consider the constant coefficient linear system

$$u_t + Au_x = 0 \qquad \begin{pmatrix} F(u) = Au \\ F'(u) = DF(u) = A \end{pmatrix}$$
$$u(x,0) = u^0(x)$$

diagonalize A: $\quad A = Q\Lambda Q^{-1}, \quad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}, \quad Q = (q_1, \dots, q_m)$

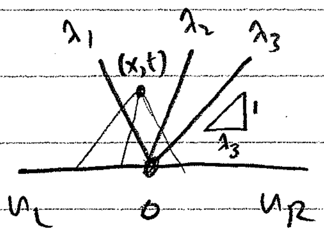
$$Q^{-1} = \begin{pmatrix} r_1^T \\ \vdots \\ r_m^T \end{pmatrix}, \quad r_i^T q_j = \delta_{ij}$$
(dual basis)

now solve de-coupled scalar equations

$$w = Q^{-1}u$$
$$w_t + \Lambda w_x = 0$$
$$w(x,0) = w^0(x) = Q^{-1}u^0(x)$$
$$w_i(x,t) = w_i^0(x - \lambda_i t) = r_i^T u^0(x - \lambda_i t)$$
$$u(x,t) = Qw(x,t) = \sum_i q_i r_i^T u^0(x - \lambda_i t)$$

same formula works if u^0 is discontinuous: $u^0(x) = \begin{cases} u_L & x<0 \\ u_R & x>0 \end{cases}$

$$u(x,t) = \sum_{i=1}^m q_i r_i^T u^0(x - \lambda_i t)$$
$$= \sum_{i:\, x<\lambda_i t} q_i r_i^T u_L + \sum_{i:\, x>\lambda_i t} q_i r_i^T u_R$$



note that $u(x,t)$ jumps by $\alpha_i q_i$, where $\alpha_i = r_i^T(u_R - u_L)$, when x crosses $\lambda_i t$.

also note that there are no rarefaction waves in the linear case.

In particular,

$$u^{\Psi} = u(0,t) = \sum_{i:\lambda_i>0} q_i r_i^T u_L + \sum_{i:\lambda_i<0} q_i r_i^T u_R$$

$$F(u^{\Psi}) = Au^{\Psi} = \sum_{i:\lambda_i>0} \lambda_i q_i r_i^T u_L + \sum_{i:\lambda_i<0} \lambda_i q_i r_i^T u_R$$

$Aq_i = \lambda_i q_i$

$$= A^+ u_L + A^- u_R$$

notation: $\lambda^+ = \max(\lambda, 0)$, $\lambda^- = \min(\lambda, 0)$

$$\Lambda^+ = \begin{pmatrix} \lambda_1^+ & & \\ & \ddots & \\ & & \lambda_m^+ \end{pmatrix}, \quad \Lambda^- = \begin{pmatrix} \lambda_1^- & & \\ & \ddots & \\ & & \lambda_m^- \end{pmatrix}$$

$A^+ = Q\Lambda^+ Q^{-1}$ // responsible for right-moving waves

$A^- = Q\Lambda^- Q^{-1}$ " " " left-moving "

$A = A^+ + A^-$, $|A| = A^+ - A^- = Q|\Lambda|Q^{-1}$

Godunov reduces to upwind in this linear case:

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\Big[\big(A^+ u_{j-1} + A^- u_j\big) - \big(A^+ u_j + A^- u_{j+1}\big)\Big]$$

$$= u_j^n - \frac{\Delta t}{\Delta x}\Big[A^+ \Delta u_{j-\frac{1}{2}} + A^- \Delta u_{j+\frac{1}{2}}\Big]$$

upwind: $u_t + au_x = 0$

$a>0$: $u_j^{n+1} = u_j^n - a\nu(u_j^n - u_{j-1}^n)$

$a<0$: $u_j^{n+1} = u_j^n - a\nu(u_{j+1}^n - u_j^n)$

either way, $u_j^{n+1} = u_j^n - \nu\left(a^+ \Delta u_{j-\frac{1}{2}}^n + a^- \Delta u_{j+\frac{1}{2}}^n\right)$

High resolution methods

try to improve order of accuracy in smooth regions while keeping shocks sharp and avoiding oscillations

key ideas: rewrite Lax-Wendroff flux

$$G^n_{i-\frac{1}{2}} = \underbrace{\tfrac{1}{2} A\left(u^n_{i-1} + u^n_i\right)}_{\text{unstable centered flux}} - \tfrac{1}{2}\frac{\Delta t}{\Delta x} A^2\left(u^n_i - u^n_{i-1}\right)$$

as

$$G^n_{i-\frac{1}{2}} = \underbrace{\left(A^+ u^n_{i-1} + A^- u^n_i\right)}_{\text{stable upwind flux}} + \underbrace{\tfrac{1}{2}|A|\left(I - \frac{\Delta t}{\Delta x}|A|\right)\left(u^n_i - u^n_{i-1}\right)}_{2^{nd}\text{ order correction}}$$

here we used $|A| = A^+ - A^-$ (just expand it all out and collect terms)

now we choose a flux limiter function $\phi(\theta)$

with the idea that $\phi(\theta) = 1 \Rightarrow 2^{nd}$ order

$\phi(\theta) = 0 \rightarrow 1^{st}$ order

scheme: (scalar case) multiply correction term by $\phi\left(\theta^n_{i-1/2}\right)$

$$\theta^n_{i-\frac{1}{2}} = \frac{\Delta u^n_{I-1/2}}{\Delta u^n_{i-1/2}} \qquad I = \begin{cases} i-1 & a_{i-\frac{1}{2}} > 0 \\ i+1 & a_{i-\frac{1}{2}} < 0 \end{cases} \quad \text{upwind direction}$$

in smooth regions, $\theta \simeq 1$; near shocks, θ can have any value.

upwind method: $\phi(\theta) = 0$

Lax-Wendroff method: $\phi(\theta) = 1$

$$\operatorname{minmod}(a,b) = \begin{cases} 0 & ab \le 0 \\ a & |a|<|b|,\ ab>0 \\ b & |b|<|a|,\ ab>0 \end{cases}$$

minmod ": $\phi(\theta) = \operatorname{minmod}(1,\theta)$

superbee ": $\phi(\theta) = \max(0, \min(1, 2\theta), \min(2,\theta))$

MC : $\phi(\theta) = \max(0, \min((1+\theta)/2, 2, 2\theta))$

the latter 3 schemes are TVD = total variation diminishing (no oscillations)

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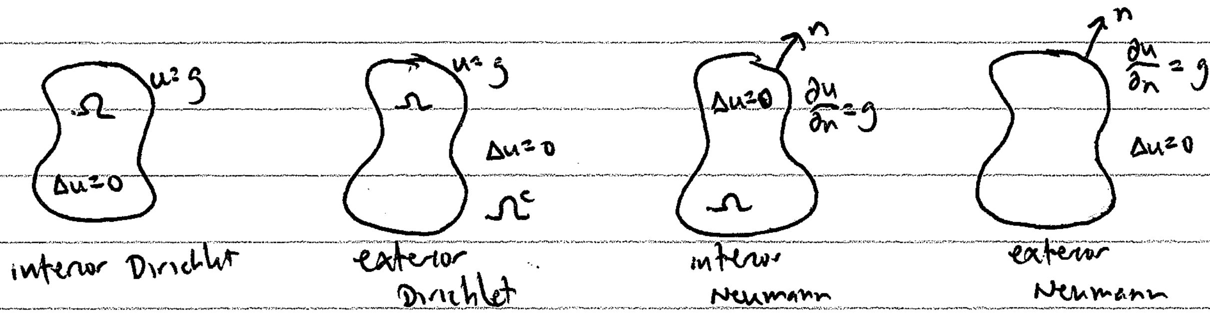
(continuation from 228A notes)

last time: double-layer $u(\vec{x}) = \int_\Gamma -\frac{\partial N}{\partial n_\xi}(\vec{x},\vec{\xi})\,\phi(\xi)\,ds_\xi$

single layer $u(\vec{x}) = \int_\Gamma N(\vec{x},\vec{\xi})\,\phi(\vec{\xi})\,ds_\xi$

4 types of B.C.'s:



the integral equations of potential theory are

interior Dirichlet	$(\frac{1}{2}\mathbb{I} + \mathbb{K})\,\phi = g$	adjoint pairs of equations
exterior Dirichlet	$(-\frac{1}{2}\mathbb{I} + \mathbb{K})\,\phi = g$	
interior Neumann	$(-\frac{1}{2}\mathbb{I} + \mathbb{K}^*)\,\phi = -g$	
exterior Neumann	$(\frac{1}{2}\mathbb{I} + \mathbb{K}^*)\,\phi = -g$	

$g = \frac{\partial u}{\partial n} = \nabla u \cdot n$, n = outward normal from Ω in both cases

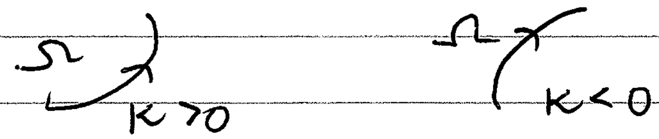
$\mathbb{K}\phi(\vec{x}) = \int_\Gamma K(\vec{x},\vec{\xi})\,\phi(\vec{\xi})\,ds$ $\qquad$ $\mathbb{K}^*\phi(\vec{x}) = \int_\Gamma K(\vec{\xi},\vec{x})\,\phi(\vec{\xi})\,ds$

order reversed (it's like a transpose)

$$K(\vec{x},\vec{\xi}) = \begin{cases} \frac{1}{2\pi}\frac{d\theta}{ds}(\vec{x},\vec{\xi}) & \vec{x},\vec{\xi}\in\Gamma,\ \vec{x}\neq\vec{\xi} \\ \frac{1}{4\pi}\varkappa(\vec{x}) & \vec{x}=\vec{\xi}\in\Gamma \end{cases}$$

$$\frac{d\theta}{ds} = \frac{(\xi-x)\eta' - (\eta-y)\xi'}{(\xi-x)^2+(\eta-y)^2}\cdot\frac{1}{\sqrt{(\xi')^2+(\eta')^2}}$$

$$\varkappa = \frac{\xi'\eta'' - \eta'\xi''}{[(\xi')^2+(\eta')^2]^{3/2}} = \text{curvature}$$



$\mathbb{K}$ and $\mathbb{K}^*$ are adjoints in $L^2(\Gamma)$

$$\langle \mathbb{K}\phi,\psi\rangle = \langle\phi,\mathbb{K}^*\psi\rangle \qquad \forall\ \phi,\psi\in L^2(\Gamma)$$

$$\langle\phi,\psi\rangle = \int_\Gamma \phi(\vec{x})\,\overline{\psi(\vec{x})}\,ds$$

they are also both <u>compact</u> operators (almost finite rank)

<u>Fredholm alternative:</u> Suppose $\mathbb{A}: L^2(\Gamma)\to L^2(\Gamma)$ has the form $\mathbb{A} = \alpha\mathbb{I} + \mathbb{K}$, $\alpha\in\mathbb{C}$, $\alpha\neq 0$, $\mathbb{K}$ compact

then either:

(1) $\mathbb{A}\phi = g$ and $\mathbb{A}^*\psi = \gamma$ have unique solutions for all $\phi,\psi\in L^2(\Gamma)$

– or –

(2) $\mathbb{A}\phi = g$ is solvable iff $\langle g,\psi\rangle = 0 \quad \forall\,\psi\in\mathcal{N}(\mathbb{A}^*)$
and $\mathbb{A}^*\psi = \gamma$ is solvable iff $\langle\gamma,\phi\rangle = 0 \quad \forall\,\phi\in\mathcal{N}(\mathbb{A})$

and these nullspaces have the same (finite) dimension the same thing happens in finite dimensions

example: $A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $A^* = \begin{pmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $N(A^*) = \text{span}\{w\}$, $w = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

$N(A) = \text{span}(e_3)$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Is there a solution of $Ax = b$, $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$? yes, $b^T w = 0$

- - - - - - - - - - - - , $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$? no, $b^T w = 1 \neq 0$

The Fredholm alternative reduces the question of solvability (For which g is there a solution?) to one of uniqueness (if $g = 0$, how many solutions are there?). But, this second question pertains to the adjoint problem

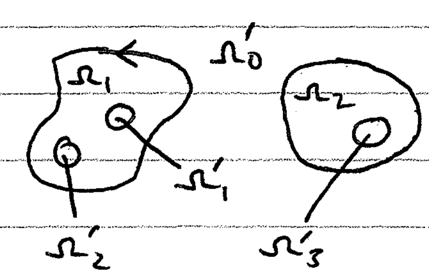
(interior Dirichlet $\longleftrightarrow$ exterior Neumann
" Neumann $\longleftrightarrow$ exterior Dirichlet)

Dimensions of Kernels. (Following Folland's PDE book)

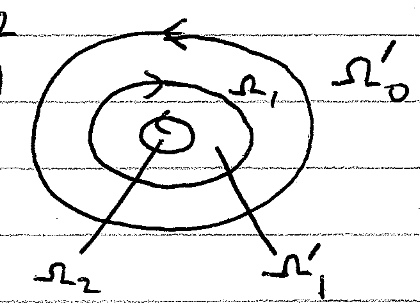
Suppose Ω has m connected components $\Omega_1, \dots, \Omega_m$
Ω' $m'+1$ $\Omega'_0, \Omega'_1, \dots, \Omega'_{m'}$

examples:

$m=2$
$m'=3$:



$m=2$
$m'=1$



Let $\phi_i(x) = \begin{cases} 1 & x \in \partial\Omega_i \\ 0 & \text{o.w.} \end{cases}$ $\quad i = 1, \dots, m$

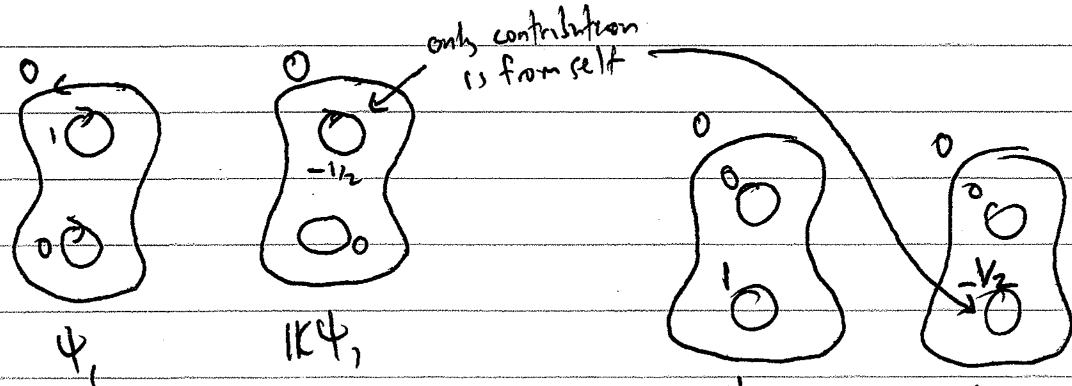
$\psi_i(x) = \begin{cases} 1 & x \in \partial\Omega_i' \\ 0 & \text{o.w.} \end{cases}$ $\quad i = 1, \dots, m'$ $\quad$ (i = 0 is excluded on purpose)

<u>claim</u>: $V = \ker\left(-\frac{1}{2}\mathbb{I} + \mathbb{K}\right) = \operatorname{span}\{\phi_i\}$

$W = \ker\left(\frac{1}{2}\mathbb{I} + \mathbb{K}\right) = \operatorname{span}\{\psi_i\}$

<u>reason</u> (in 2D): winding numbers cancel as you cross boundaries.

ϕ_1 $\qquad$ $\mathbb{K}\phi_1$



ψ_1 $\qquad$ $\mathbb{K}\psi_1$ $\qquad$ ψ_2 $\qquad$ $\mathbb{K}\psi_2$

so $\dim V \geq m$, $\dim W' \geq m'$. want to show $=$.

Let $V_1 = \ker\left(-\frac{1}{2}\mathbb{I} + \mathbb{K}^*\right)$, $W_1 = \ker\left(\frac{1}{2}\mathbb{I} + \mathbb{K}^*\right)$

by F.A., $\dim V_1 = \dim V$, $\dim W_1 = \dim W$.

<u>Claim</u>: $\dim V_1 \leq m$, $\dim W_1 \leq m'$.

if $\phi \in V_1$, then $u(\vec{x}) = \int N(\vec{x}, \vec{\xi})\, \phi(\vec{\xi})\, ds$

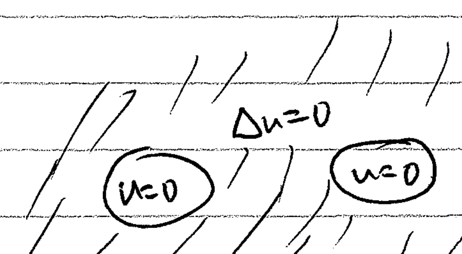
solves the interior Neumann problem $\begin{cases} \Delta u = 0 \text{ in } \Omega \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \end{cases}$

so u is constant on each connected component of Ω.

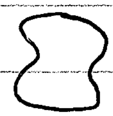
Thus, we have a linear mapping β from $\phi \in V_1$ to $\beta(\phi) = \left(u|_{\Omega_1}, \dots, u|_{\Omega_m}\right) \in \mathbb{R}^m$.

This mapping is injective (in 3d) since single layer potentials are continuous across $\partial\Omega$, and if $u = 0$ on each Ω_i, then u solves the exterior Dirichlet problem with 0 bdry conditions. In 3d, this implies $u = 0$ on Ω' as well, so $\phi = \frac{\partial u^-}{\partial n} - \frac{\partial u^+}{\partial n} = 0$.

In 2d, you instead show that $\ker(\beta)$ is one-dimensional and $\operatorname{ran}(\beta)$ has codimension 1.



complex variable methods in 2d potential theory



$\Delta u = 0$ in Ω

$u = g$ on Γ

$$u(\vec{x}) = \int_\Gamma -\frac{\partial N}{\partial n_\xi}(\vec{x},\vec{\xi})\, \mu(\vec{\xi})\, ds_\xi$$

$\uparrow$ using μ instead of ϕ today

$$\left[\begin{array}{l} \vec{x} = (x,y) \to z = x+iy \\ \vec{\xi} = (\xi,\eta) \to \zeta = \xi + i\eta \end{array}\right.$$

$\leftarrow$ curve parametrized by $\zeta(\alpha)$, $0 \le \alpha \le a$

$$-\frac{\partial N}{\partial n_\xi} ds = \frac{1}{2\pi}\frac{d\theta}{d\alpha} d\alpha = \frac{1}{2\pi}\frac{(\xi(\alpha)-x,\eta(\alpha)-y)\cdot(\eta'(\alpha),-\xi'(\alpha))}{(\xi-x)^2+(\eta-y)^2} d\alpha$$

$$\frac{(a,b)\cdot(y,-x)}{a^2+b^2} = \frac{ay-bx}{a^2+b^2} = \mathrm{Im}\left(\frac{(a-ib)(x+iy)}{a^2+b^2}\right) = \mathrm{Im}\left(\frac{x+iy}{a+ib}\right)$$

$$-\frac{\partial N}{\partial n_\xi} ds = \mathrm{Im}\left\{\frac{1}{2\pi}\frac{\zeta'(\alpha)}{\zeta(\alpha)-z} d\alpha\right\} = \mathrm{Re}\left\{\frac{1}{2\pi i}\frac{\zeta'(\alpha)}{\zeta(\alpha)-z} d\alpha\right\}$$

double layer potential becomes

$$u(z) = \mathrm{Re}\left\{\frac{1}{2\pi i}\int_0^a \frac{\zeta'(\alpha)}{\zeta(\alpha)-z}\mu(\alpha)\,d\alpha\right\}$$

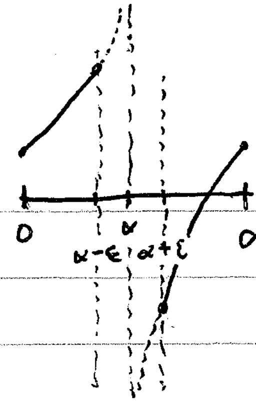
$$U(z) = u(z) + iv(z) = \frac{1}{2\pi i}\int_0^a \frac{\zeta'(\alpha)}{\zeta(\alpha)-z}\mu(\alpha)\,d\alpha$$

Cauchy Integral ($d\zeta = \zeta'(\alpha)d\alpha$)

When $z = \zeta(\alpha)$ is on the boundary, this is a <u>singular integral</u>, which has to be interpreted in the Principal Value sense:

$$U(\zeta(\alpha)) = \frac{1}{2\pi i} ⨍_0^a \frac{\zeta'(\beta)}{\zeta(\beta)-\zeta(\alpha)}\mu(\beta)\,d\beta$$

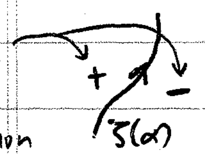
here $$⨍_0^a \cdots = \lim_{\varepsilon\to 0}\left(\int_0^{\alpha-\varepsilon} \cdots + \int_{\alpha+\varepsilon}^{a} \cdots\right)$$



remove symmetric interval $[\alpha-\varepsilon, \alpha+\varepsilon]$ from integration domain

Plemelj formula for Cauchy integrals:

note change in sign convention compared to 228A notes



$$U(\zeta(\alpha)^{\pm}) = \pm\frac{1}{2}\mu(\alpha) + \frac{1}{2\pi i}⨍_0^a \frac{\zeta'(\beta)}{\zeta(\beta)-\zeta(\alpha)}\mu(\beta)\,d\beta$$

special case: $\zeta(\alpha)=\alpha$, $\alpha\in\mathbb{R}$ (real line instead of periodic curve)

$$U(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mu(\alpha)}{\alpha - z}\,d\alpha \quad , \quad \mathrm{Re}\{z\} > 0$$

$$U(x^{\pm}) = \pm\frac{1}{2}\mu(x) + \frac{1}{2\pi i}⨍_{-\infty}^{\infty} \frac{\mu(\beta)}{\beta - x}\,d\beta$$

Hilbert transform: given $f(x) = \mathrm{Re}\{U(x^+)\}$, return $Hf(x) = \mathrm{Im}\{U(x^+)\}$

formula: $$Hf(x) = \frac{1}{\pi}⨍_{-\infty}^{\infty} \frac{f(\beta)}{x-\beta}\,d\beta \qquad \left(\begin{array}{l}\text{used } \mu(x) = 2f(x) \\ \frac{1}{i}\frac{1}{\beta-x} = \frac{i}{x-\beta}\end{array}\right)$$

when $f(x)$ is periodic, $⨍_{-\infty}^{\infty} \cdots$ means $\lim_{\substack{\varepsilon\to 0\\ N\to\infty}}\left(\int_{-N}^{x-\varepsilon} \cdots + \int_{x+\varepsilon}^{N} \cdots\right)$

write $f(x) = \sum_k \hat{f}_k e^{ikx}$, where $\hat{f}_{-k} = \overline{\hat{f}_k}$

$(z = x+iy)$

the extension $U(z) = \sum_k \hat{f}_k e^{ikz}$ is not bounded as $y\to\infty$

but $U(z) = \hat{f}_0 + 2\sum_{k=1}^{\infty} \hat{f}_k e^{ikz}$ is fine (since $e^{ik(x+iy)} = \underbrace{e^{-ky}}_{\to 0 \text{ as } y\to\infty} e^{ikx}$)

$$\mathrm{Re}\{U(x)\} = \hat{f}_0 + \sum_{k=1}^{\infty}\left[\hat{f}_k e^{ikz} + \bar{\hat{f}}_k e^{-ikz}\right] = f(x)$$

$$\mathrm{Im}\{U(x)\} = \sum_{k=1}^{\infty}\left[-i\hat{f}_k e^{ikz} + i\bar{\hat{f}}_k e^{-ikz}\right] = Hf(x)$$

So $\widehat{Hf}_k = \begin{cases} -i\hat{f}_k & k>0 \\ 0 & k=0 \\ i\hat{f}_k & k<0 \end{cases}$, Here we used $\mathrm{Re}\, w = \frac{w+\bar{w}}{2}$, $\mathrm{Im}\, w = \frac{w-\bar{w}}{2i}$

Alternatively, when $f(x)$ is periodic, we can write

$$Hf(x) = \frac{1}{\pi}\int_0^{2\pi} \underbrace{\mathrm{P.V.}\sum_{k=-\infty}^{\infty}}_{\text{principal value sum}} \frac{f(\beta)}{x-(\beta+2\pi k)}\, d\beta$$

principal value sum $\mathrm{P.V.}\sum_k a_k = a_0 + \sum_{k=1}^{\infty}[a_k + a_{-k}]$

Euler: $$\sin z = z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{\pi^2 k^2}\right) = z\ \mathrm{P.V.}\prod_{k\neq 0}\left(1-\frac{z}{\pi k}\right)$$

$$\log \sin z = \log z + \mathrm{P.V.}\sum_{k\neq 0}\log\left(1-\frac{z}{\pi k}\right)$$

$\frac{d}{dz}$

$$\cot z = \frac{1}{z} + \mathrm{PV}\sum_{k\neq 0}\frac{1}{z-\pi k} = \mathrm{PV}\sum_k \frac{1}{z-\pi k}$$

$$\frac{1}{2}\cot\frac{z}{2} = \mathrm{PV}\sum_k \frac{1}{z-2\pi k}$$

so $$Hf(x) = \frac{1}{\pi}\int_0^{2\pi}\frac{f(\beta)}{2}\cot\left(\frac{x-\beta}{2}\right)d\beta$$

We can use this form of the Hilbert transform to regularize a general Cauchy integral:

$$U(\zeta(\alpha)^{\pm}) = \pm\frac{1}{2}\mu(\alpha) + \frac{1}{2\pi i}\int_0^a \frac{\zeta'(\beta)}{\zeta(\beta)-\zeta(\alpha)}\mu(\beta)\,d\beta$$

suppose $a = 2\pi$ for simplicity. Then

$$\frac{1}{2\pi i}\int_0^{2\pi} \cdots \, d\beta = \frac{1}{2\pi i}\int_0^{2\pi}\left[\frac{\zeta'(\beta)}{\zeta(\beta)-\zeta(\alpha)} - \frac{1}{2}\cot\left(\frac{\beta-\alpha}{2}\right)\right]\mu(\beta)\,d\beta$$

$$+ \frac{1}{2\pi i}\int_0^{2\pi}\frac{1}{2}\cot\left(\frac{\beta-\alpha}{2}\right)\mu(\beta)\,d\beta$$

just a Riemann integral now since $\lim_{\beta\to\alpha}[\cdots] = \frac{\zeta''(\alpha)}{2\zeta'(\alpha)}$

(the singular leading terms cancel)

still a principal value integral, but a particularly nice one (it equals $\frac{i}{2}H\mu(\alpha)$)

We can evaluate it using the FFT!

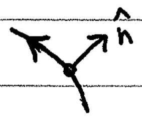
When we take the real part of $U(\zeta(\alpha)^{\pm})$, the regularizing terms disappear (as they are imaginary) and we have

$$\frac{1}{2}\mu(\alpha) + \frac{1}{2\pi}\int_0^{2\pi} K(\alpha,\beta)\mu(\beta)\,d\beta = g(\alpha)$$

$$K(\alpha,\beta) = \mathrm{Re}\left\{\frac{1}{i}\frac{\zeta'(\beta)}{\zeta(\beta)-\zeta(\alpha)}\right\} = \begin{cases} \mathrm{Im}\left\{\dfrac{\zeta'(\beta)}{\zeta(\beta)-\zeta(\alpha)}\right\} & \alpha\neq\beta \\[2ex] \mathrm{Im}\left\{\dfrac{\zeta''(\beta)}{2\zeta'(\beta)}\right\} & \alpha=\beta \end{cases}$$

matrix form: $\left(\frac{1}{2}\mathbb{I} + \mathbb{K}\right)\mu = g$, $\mathbb{K}_{kj} = \frac{1}{N}K(\alpha_k,\beta_j)$, $\alpha_k = \frac{2\pi k}{N}$, $\beta_j = \frac{2\pi j}{N}$

Next we want to compute the normal derivative of u on the boundary. For any function $u(z)$, we have



$$\frac{\partial u}{\partial n} = (u_x, u_y) \cdot \frac{(\eta', -\xi')}{|\zeta'|} = \operatorname{Im}\left\{ \frac{(u_x - i u_y)(\xi' + i\eta')}{|\zeta'|} \right\}$$

If $U(z) = u(z) + i v(z)$ is analytic, then

$$u_x = v_y, \quad u_y = -v_x \qquad \text{(Cauchy-Riemann equations)}$$

so $$\frac{\partial u}{\partial n} = \operatorname{Im}\left\{ \frac{(u_x + i v_x)(\zeta')}{|\zeta'|} \right\} = \operatorname{Im}\left\{ \frac{U_z \zeta'}{|\zeta'|} \right\}$$

this formula could be used for the exact solution in the homework:

$$U(z) = 2\log\big(z - (i-2)\big) \Rightarrow \frac{\partial u}{\partial n}(\zeta(\alpha)) = \operatorname{Im}\left\{ \frac{2}{z-(i-2)} \cdot \frac{\zeta'(\alpha)}{|\zeta'(\alpha)|} \right\}$$

It can also be used for the numerical solution:

$$U(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\zeta'(\alpha)}{\zeta(\alpha) - z}\, \mu(\alpha)\, d\alpha$$

$$\frac{\partial u}{\partial n}(z,\alpha) = \nabla u(z) \cdot n_{\zeta(\alpha)} = \operatorname{Im}\left\{ \left(\frac{1}{2\pi i} \int_0^{2\pi} \frac{\zeta'(\beta)}{(\zeta(\beta) - z)^2}\, \mu(\beta)\, d\beta \right) \frac{\zeta'(\alpha)}{|\zeta'(\alpha)|} \right\}$$

note that $\dfrac{\zeta'(\beta)}{(\zeta(\beta) - z)^2} = -\dfrac{\partial}{\partial\beta} \dfrac{1}{\zeta(\beta) - z}$ so we can integrate by parts to get

$$\frac{\partial u}{\partial n}(z,\alpha) = \operatorname{Im}\left\{ \left(\frac{1}{2\pi i} \int_0^{2\pi} \frac{\mu'(\beta)}{\zeta(\beta) - z}\, d\beta \right) \frac{\zeta'(\alpha)}{|\zeta'(\alpha)|} \right\}$$

We now define $\frac{\partial u}{\partial n}(\zeta(\alpha)^{\pm}) = \lim_{z \to \zeta(\alpha)^{\pm}} \frac{\partial u}{\partial n}(z,\alpha)$, which gives

$$|\zeta'(\alpha)| \frac{\partial u}{\partial n}(\zeta(\alpha)^{\pm}) = \mathrm{Im}\left\{ \lim_{z \to \zeta(\alpha)^{\pm}} \frac{1}{2\pi i} \int_0^{2\pi} \frac{\zeta'(\alpha)}{\zeta(\beta) - z} \mu'(\beta)\, d\beta \right\}$$

this limit can be evaluated by writing $\frac{\zeta'(\alpha)}{\zeta(\beta)-z} = \underbrace{\frac{\zeta'(\alpha) - \zeta'(\beta)}{\zeta(\beta) - z}}_{\text{not singular as } z \to \zeta(\alpha)} + \underbrace{\frac{\zeta'(\beta)}{\zeta(\beta)-z}}_{\text{use Plemelj}}$

result:

$$|\zeta'(\alpha)| \frac{\partial u}{\partial n}(\zeta(\alpha)^{\pm}) = \mathrm{Im}\left\{ \pm \frac{1}{2}\mu'(\alpha) + \frac{1}{2\pi i} ⨍_0^{2\pi} \frac{\zeta'(\alpha)}{\zeta(\beta) - \zeta(\alpha)} \mu'(\beta)\, d\beta \right\}$$

so $\frac{\partial u}{\partial n}(z,\alpha)$ is actually continuous (in z) across Γ since the jump term is real.

We use our regularization trick to evaluate the principal value integral:

$$|\zeta'(\alpha)| \frac{\partial u}{\partial n} = \mathrm{Im}\left\{ \frac{1}{2\pi i} \int_0^{2\pi} \left[\frac{\zeta'(\alpha)}{\zeta(\beta) - \zeta(\alpha)} - \frac{1}{2}\cot\left(\frac{\beta - \alpha}{2}\right) \right] \mu'(\beta)\, d\beta \right\}$$

$$+ \mathrm{Im}\left\{ \frac{1}{2\pi i} ⨍_0^{2\pi} \frac{1}{2}\cot\left(\frac{\beta-\alpha}{2}\right) \mu'(\beta)\, d\beta \right\}$$

or $$|\zeta'(\alpha)| \frac{\partial u}{\partial n}(\zeta(\alpha)) = \frac{1}{2} H[\mu'](\alpha) + \frac{1}{2\pi} \int_0^{2\pi} G(\alpha,\beta)\, \mu'(\beta)\, d\beta$$

$$G(\alpha,\beta) = \begin{cases} \mathrm{Re}\left\{ \frac{\zeta'(\alpha)}{\zeta(\alpha) - \zeta(\beta)} - \frac{1}{2}\cot\left(\frac{\alpha-\beta}{2}\right) \right\} & \alpha \neq \beta \\ \mathrm{Re}\left\{ \frac{\zeta''(\alpha)}{2\zeta'(\alpha)} \right\} & \alpha = \beta \end{cases}$$

in matrix form: $\frac{\partial u}{\partial n} = \left(\frac{1}{2} H D \mu + G D \mu \right)$ divided by $|\zeta'(\alpha)|$, $\quad G_{kj} = \frac{1}{N} G(\alpha_k, \beta_j)$

The FFT can be used to apply H and D without actually forming these matrices:

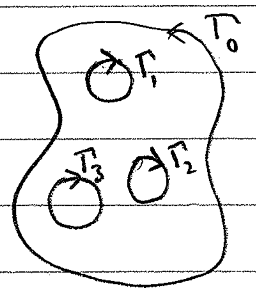
$$Df = \mathcal{F}^{-1}\operatorname{diag}(ih^{-1}\xi_k)\mathcal{F}f, \qquad Hf = \mathcal{F}^{-1}\operatorname{diag}(-i\,\operatorname{sgn}(k))\mathcal{F}f$$

$$\xi_k = \frac{2\pi}{N}\begin{cases} k-1 & 1 \le k \le \frac{N}{2} \\ k-1-N & \frac{N}{2}+2 \le k \le N \\ 0 & k = N/2+1 \end{cases}, \qquad \operatorname{sgn}(k) = \begin{cases} 1 & 2 \le k \le \frac{N}{2} \\ -1 & \frac{N}{2}+2 \le k \le N \\ 0 & k=1 \text{ or } k = \frac{N}{2}+1 \end{cases}$$

Nyquist

Multiply-connected domains

Suppose Ω has one connected component and $\Omega' = \mathbb{R}^2 \setminus \bar{\Omega}$ has m' connected components.



$m'=3$ example

The operator $\frac{1}{2}\mathbb{I} + \mathbb{K}$ has a kernel of dimension m'. Specifically, the functions

$$\mathbb{1}_k(x) = \begin{cases} 1 & x \in \Gamma_k \\ 0 & x \in \Gamma_j,\ j \ne k \end{cases}$$

are a basis for $\ker(\frac{1}{2}\mathbb{I} + \mathbb{K})$

By the Fredholm alternative, $\frac{1}{2}\mathbb{I} + \mathbb{K}^*$ also has a kernel, V^*, of dimension m', and

$$\mathbb{B}\mu = \left(\frac{1}{2}\mathbb{I} + \mathbb{K}\right)\mu = g$$

is solvable iff $\langle g, \varphi\rangle = 0 \quad \forall \varphi \in \ker\left(\frac{1}{2}\mathbb{I} + \mathbb{K}^*\right)$

The solution is to modify the layer potential so that the resulting integral equation takes the form

$$\tilde{\mathbb{B}}\mu = g, \qquad \tilde{\mathbb{B}} = \mathbb{B} + \sum_{k=1}^{m'} \langle \cdot, 1_k \rangle \varphi_k$$

where $\mathrm{span}\{\varphi_1, \ldots, \varphi_{m'}\}$ is any complement of $\underbrace{(V^*)^\perp}_{\text{range of } \mathbb{B}.}$ in $L^2(\Gamma)$.

I claim the functions $\varphi_j(\vec{x}) = \log|\vec{x} - \vec{a}_j|$, $1 \le j \le m'$, $\vec{x} \in \Gamma$ work, where $\vec{a}_j$ is an arbitrary point inside T_j. Suppose to the contrary that a nonzero linear combination $g = \sum_j \alpha_j \varphi_j$ belongs to $(V^*)^\perp$. Then $g \in \mathrm{ran}(\mathbb{B})$, so the solution of the Dirichlet problem $\left\{ \begin{array}{l} \Delta u = 0 \text{ in } \Omega \\ u = g \text{ on } \Gamma \end{array} \right\}$ can be represented using a double-layer potential. But the solution of this problem is $u(\vec{x}) = \sum_j \alpha_j \log|\vec{x} - \vec{a}_j|$ for $\vec{x} \in \Omega$. Since the normal derivative $\frac{\partial u}{\partial n}$ of a double layer potential is continuous across each T_i, the function $u(\vec{x})$ satisfies the Neumann problem

$$\Delta u = 0 \quad \text{in } \Omega_i'$$

$$\frac{\partial u}{\partial n} = \tilde{g}_i \quad \text{on } T_i = \partial\Omega_i'$$

where $\tilde{g}_i(\vec{x}) = \sum_j \alpha_j \frac{\partial}{\partial n} \log|\vec{x} - \vec{a}_j|$ for $\vec{x} \in T_i$.

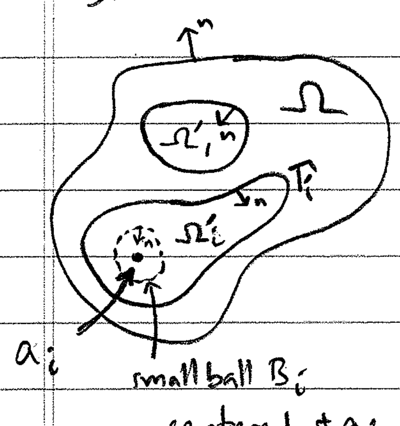
In other words, the data for the Neumann problem in the ith hole comes from the normal derivative of the solution just outside the hole.

But the ith Neumann problem has a solution iff $\int \tilde{g}_i \, ds = 0$.

This follows from the divergence theorem:

$$\int_{\Gamma_i} \tilde{g}_i \, ds = \int_{\Gamma_i} \nabla u \cdot n \, ds = \int_{\Omega'_i} \underbrace{\nabla \cdot (\nabla u)}_{0} \, dx = 0$$

Since the function $w(\vec{x}) = \sum_j \alpha_j \log|\vec{x} - a_j|$ satisfies $\Delta w = 0$ in



$\Omega'_i \setminus B_i$, we can apply the same argument to conclude

$$\int_{\Gamma_i} \underbrace{\frac{\partial w}{\partial n}}_{\tilde{g}_i} \, ds = \int_{\partial B_i} \frac{\partial w}{\partial n} \, ds = \int_0^{2\pi} -\frac{\alpha_i}{r} \, r \, d\theta = -2\pi\alpha_i$$

only $j = i$ term contributes

So all the α_i must be zero, which is a contradiction.

Going back to complex variables and parametrizing each curve Γ_i separately by its own function $\zeta_i(\alpha)$, the new representation is

$$U(z) = \sum_{j=0}^{m'} \frac{1}{2\pi i} \int_0^{2\pi} \frac{\zeta_j'(\alpha)}{\zeta_j(\alpha) - z} \mu_j(\alpha) \, d\alpha$$

$$+ \sum_{j=1}^{m'} \log(z - a_j) \cdot \underbrace{\frac{1}{2\pi} \int_0^{2\pi} \mu_j(\alpha) \, d\alpha}_{P_0 \mu_j}$$

complex logarithm

$U(z) = u(z) + i v(z)$

$u(z) = \mathrm{Re}\{U(z)\}$

P_0 is an operator that computes the average value of a function.

The integral equations we need to solve are

$$\frac{1}{2}\mu_k(\alpha) + \sum_{j=0}^{m'} \frac{1}{2\pi}\int_0^{2\pi} \left[K_{kj}(\alpha,\beta) + C_{kj}(\alpha)\,1(\beta)\right]\mu_j(\beta)\,d\beta = g_k(\alpha)$$

$$K_{kj}(\alpha,\beta) = \begin{cases} \mathrm{Im}\left\{ \dfrac{\zeta_j'(\beta)}{\zeta_j(\beta) - \zeta_k(\alpha)} \right\} & k \neq j \text{ or } \alpha \neq \beta \\[2ex] \mathrm{Im}\left\{ \dfrac{\zeta_j''(\alpha)}{2\,\zeta_j'(\alpha)} \right\} & k = j \text{ and } \alpha = \beta \end{cases}$$

$$C_{kj}(\alpha) = \begin{cases} \log|\zeta_k(\alpha) - a_j| & j = 1,\dots,m' \\ 0 & j = 0 \end{cases}$$

$$1(\beta) = 1$$

The matrix version when $m' = 2$ is

$$\left[\frac{1}{2}\begin{pmatrix} I & & \\ & I & \\ & & I \end{pmatrix} + \begin{pmatrix} K_{00} & K_{01} & K_{02} \\ K_{10} & K_{11} & K_{12} \\ K_{20} & K_{21} & K_{22} \end{pmatrix} + \begin{pmatrix} C_{00} & C_{01} & C_{02} \\ C_{10} & C_{11} & C_{12} \\ C_{20} & C_{21} & C_{22} \end{pmatrix}\right]\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$$

where $(K_{kj})_{\ell m} = \dfrac{1}{n_j} K_{kj}(\alpha_\ell, \beta_m)$

$(C_{kj})_{\ell m} = \dfrac{1}{n_j} C_{kj}(\alpha_\ell)\, 1(\beta_m)$

curve Γ_j has n_j collocation points.

(The rows of the matrix represent trapezoidal rule approximations of the corresponding integrals)

To compute the normal derivative of the solution, we use the formula

$$|\zeta_k'(\alpha)| \frac{\partial u}{\partial n}(\zeta_k(\alpha)) =$$

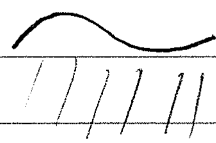
$$\frac{1}{2} H[\mu_k'](\alpha) + \sum_{j=0}^{m'} \frac{1}{2\pi} \int_0^{2\pi} G_{kj}(\alpha,\beta)\, \mu'(\beta)\, d\beta$$

$$+ \sum_{j=1}^{m'} \operatorname{Im}\left\{ \frac{\zeta_k'(\alpha)}{\zeta_k(\alpha) - a_j} \right\} P_0 \mu_j$$

where

$$G_{kj}(\alpha,\beta) = \begin{cases} \operatorname{Re}\left\{ \dfrac{\zeta_k'(\alpha)}{\zeta_k(\alpha) - \zeta_j(\beta)} - \dfrac{1}{2}\cot\left(\dfrac{\alpha-\beta}{2}\right) \right\} & k=j,\ \alpha \neq \beta \\ \operatorname{Re}\left\{ \dfrac{\zeta_k''(\alpha)}{2\zeta_k'(\alpha)} \right\} & k=j,\ \alpha=\beta \\ \operatorname{Re}\left\{ \dfrac{\zeta_k'(\alpha)}{\zeta_k(\alpha) - \zeta_j(\beta)} \right\} & k \neq j \end{cases}$$

water waves

setup:  ← free surface $\eta(x,t)$ evolving in time

← Euler equations (incompressible, irrotational, inviscid)

$$\circledast \begin{cases} \rho\left(\vec{u}_t + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla p - \rho g \hat{y} \\ \nabla\cdot\vec{u} = 0 \end{cases}$$

$\vec{u}$ = velocity
p = pressure
ρ = density
g = gravitational acceleration

assumption: vorticity is zero

→ there is a potential function ϕ such that $\vec{u} = \nabla\phi$
(assuming the domain is simply-connected)

since $\nabla\left(\frac{1}{2}\|\vec{u}\|^2\right) = \nabla\left(\frac{1}{2}u^2 + \frac{1}{2}v^2\right) = \begin{pmatrix} uu_x + vv_x \\ uu_y + vv_y \end{pmatrix}$

and $\vec{u}\cdot\nabla\vec{u} = (u\partial_x + v\partial_y)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} uu_x + vu_y \\ uv_x + vv_y \end{pmatrix}$

we see that they are equal if the flow is irrotational $\left(\nabla\times\vec{u} = (v_x - u_y)\hat{z} = 0\right)$

$\circledast$ becomes
$$\begin{cases} \nabla\left(\phi_t + \frac{1}{2}\|\nabla\phi\|^2 + \frac{p}{\rho} + gy\right) = 0 \\ \Delta\phi = 0 \end{cases}$$

so $\boxed{\phi_t + \frac{1}{2}\|\nabla\phi\|^2 + \frac{p}{\rho} + gy = C(t)}$ ← constant in space, allowed to vary in time.

unsteady Bernoulli equation

consider a particle $(x(t), y(t))$ on the free surface.

$$y(t) = \eta(x(t), t)$$
$$\dot{y} = \eta_x \dot{x} + \eta_t$$
$$v = \eta_x u + \eta_t \rightarrow \boxed{\eta_t = \phi_y - \eta_x \phi_x}$$

We also define $\Phi(x,t) = \phi(x, \eta(x,t), t)$ and check that

$$\Phi_t = \phi_y \eta_t + \phi_t$$

using the Bernoulli equation to evaluate ϕ_t, we get

$$\Phi_t = \phi_y \eta_t - \frac{1}{2}\phi_x^2 - \frac{1}{2}\phi_y^2 - \frac{P}{\rho} - gy + c(t)$$

we usually choose $c(t)$ so that $\int_0^{2\pi} \Phi(x,t)\,dx = 0$.

Final set of equations:

initial conditions: $\eta(x,0) = \eta_0(x)$, $\Phi(x,0) = \Phi_0(x)$ $(t=0)$

have to solve Laplace equation to obtain ϕ_x, ϕ_y on boundary

$$\left[\begin{array}{ll} \phi_{xx} + \phi_{yy} = 0 & (y < \eta) \\ \phi_y = 0 & (y = -\infty) \\ \phi = \Phi & (y = \eta) \end{array}\right.$$

actual evolution equations ($p = 0$ at free surface)

$$\left[\begin{array}{ll} \eta_t = \phi_y - \eta_x \phi_x & (y = \eta) \\ \Phi_t = P\left[\phi_y \eta_t - \frac{1}{2}|\nabla\phi|^2 - g\eta\right] & (y = \eta) \end{array}\right.$$

project out the mean $Pf(x) = f(x) - \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$

note that $\Phi_x = \phi_x + \eta_x \phi_x = \sqrt{1+\eta_x^2}\,\frac{\partial\phi}{\partial s}$

define $\mathcal{G}\Phi = \phi_y - \eta_x \phi_x = \sqrt{1+\eta_x^2}\,\frac{\partial\phi}{\partial n}$ ← Dirichlet-Neumann operator

so $\begin{pmatrix}\Phi_x \\ \mathcal{G}\Phi\end{pmatrix} = \begin{pmatrix} 1 & \eta_x \\ -\eta_x & 1\end{pmatrix}\begin{pmatrix}\phi_x \\ \phi_y\end{pmatrix}$ and $\begin{pmatrix}\phi_x \\ \phi_y\end{pmatrix} = \frac{1}{1+\eta_x^2}\begin{pmatrix} 1 & -\eta_x \\ \eta_x & 1\end{pmatrix}\begin{pmatrix}\Phi_x \\ \mathcal{G}\Phi\end{pmatrix}$

Energy conservation

transport theorem: if W_t is a region moving with the fluid, then

$$\frac{d}{dt}\int_{W_t} \rho f \, dV = \int_{W_t} \rho \frac{Df}{Dt} \, dV \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla$$

$f(\vec{x},t)$ a function

$$\frac{d}{dt} E_{kinetic} = \frac{d}{dt}\left[\frac{1}{2}\int_{W_t} \rho \|\vec{u}\|^2 dV\right] = \frac{1}{2}\int_{W_t} \rho \frac{D\|\vec{u}\|^2}{Dt} dV$$

$$= \int_{W_t} \rho\left(\vec{u} \cdot \frac{D\vec{u}}{Dt}\right) dV = \int_{W_t} -\left(\vec{u} \cdot \nabla(p + \rho g y)\right) dV$$

$$= \int_{W_t} -\operatorname{div}\left[(p+\rho g y)\vec{u}\right] dV = \int_{\Gamma} (\underset{\substack{\uparrow \\ 0}}{p} + \rho g \underset{\substack{\uparrow \\ \eta}}{y}) \underbrace{\vec{u}\cdot\hat{n}}_{\frac{\partial \phi}{\partial n}} \, ds$$

$$= \int_0^{2\pi} \rho g \eta \sqrt{1+\eta_x^2} \frac{\partial \phi}{\partial n} dx$$

$$= \int_0^{2\pi} \rho g \eta \eta_t \, dx = \frac{\partial}{\partial t} \underbrace{\frac{1}{2}\int_0^{2\pi} \rho g \eta^2 dx}_{\text{potential energy of fluid}}$$

Also, $\frac{1}{2}\int_{W_t} \rho \|\vec{u}\|^2 dV = \frac{1}{2}\int_{W_t} \rho \, \nabla\phi \cdot \nabla \phi \, dV$

$$= \frac{1}{2}\int_{W_t} \rho \nabla \cdot (\phi \nabla \phi) \, dV = \frac{1}{2}\int_{\Gamma} \rho \phi \frac{\partial \phi}{\partial n} ds$$

so $\boxed{E = \frac{1}{2}\int_0^{2\pi} \rho \phi \frac{\partial \phi}{\partial n} ds + \frac{1}{2}\int_0^{2\pi} \rho g \eta^2 dx}$ is conserved

good to monitor this to make sure solution is well resolved.

Also note that $\frac{\partial}{\partial t}\int_0^{2\pi} \eta \, dx = \int_0^{2\pi} \eta_t \, dx = \int_{\Gamma} \frac{\partial \phi}{\partial n} ds = \int_W \nabla \cdot \nabla \phi \, dV = 0$

For traveling waves, we assume $\begin{cases} \eta(x,t) = \eta(x-ct,0) \\ \phi(x,y,t) = \phi(x-ct,y,0) \end{cases}$

$$\frac{\partial}{\partial t}\eta(x,t) = -c\eta_x(x-ct,0)$$
$$\frac{\partial}{\partial t}\phi(x,y,t) = -c\phi_x(x-ct,y,0)$$

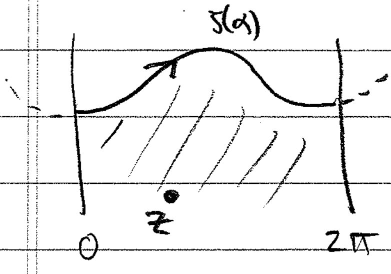
$$\eta_t = \phi_y - \eta_x\phi_x \longrightarrow (\phi_x - c)\eta_x = \phi_y$$
$$\phi_t + \tfrac{1}{2}\|\nabla\phi\|^2 + g\eta = C(t) \longrightarrow \left(\tfrac{1}{2}\phi_x - c\right)\phi_x + \tfrac{1}{2}\phi_y^2 + g\eta = 0$$

LHS is independent of t, so $C(t) = \text{const}$.
in fact, $C(t)$ may be taken to be zero by choosing the mean value of η correctly. (In numerical simulations, I find that setting $\int \eta\,dx = 0$ causes $C(t)$ to be zero, but I'm not sure how to prove that this is always the case)

As shown earlier, $\begin{pmatrix}\phi_x \\ \phi_y\end{pmatrix} = \dfrac{1}{1+\eta_x^2}\begin{pmatrix}1 & -\eta_x \\ \eta_x & 1\end{pmatrix}\begin{pmatrix}\Phi_x \\ G\Phi\end{pmatrix}$

So everything boils down to computing $G\Phi = \sqrt{1+\eta_x^2}\,\dfrac{\partial\phi}{\partial n}$

For this, we use a double-layer potential, but now on a 2π-periodic domain.



$$\zeta(\alpha) = \alpha + i\eta(\alpha)$$

(drop t in the notation for η)

$$\zeta'(\alpha) = 1 + i\eta'(\alpha)$$

In the past, we put Ω to the left of the curve, but here it's more natural to parametrize ζ so Ω is to the right. Rather than think of this as an exterior problem, we'll insert a minus sign in front of $\zeta'(\alpha)$:

$$\phi(z) = \mathrm{Re}\left\{ \frac{1}{2\pi i} \fint_{-\infty}^{\infty} \frac{-\zeta'(\alpha)}{\zeta(\alpha) - z} \mu(\alpha)\, d\alpha \right\}$$

we can eliminate the principal value integral by summing over periodic images:

$$\sum_k \frac{1}{(\zeta(\alpha) + 2\pi k) - z} = \frac{1}{2}\cot\left(\frac{\zeta(\alpha) - z}{2}\right)$$

$$\phi(z) = \mathrm{Re}\left\{ \frac{1}{2\pi i} \int_0^{2\pi} \frac{\zeta'(\alpha)}{2} \cot\left(\frac{z - \zeta(\alpha)}{2}\right) \mu(\alpha)\, d\alpha \right\}$$

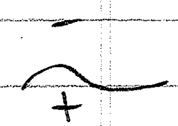
moved minus sign in

the quantity inside braces is the complex velocity potential,

$$w = \phi + i\psi$$

ϕ = scalar velocity potential
ψ = stream function

The Plemelj formula becomes



$$w(\zeta(\alpha)^+) = \frac{1}{2}\mu(\alpha) + \frac{1}{2\pi i}\fint_0^{2\pi} \frac{\zeta'(\beta)}{2}\cot\left(\frac{\zeta(\alpha)-\zeta(\beta)}{2}\right)\mu(\beta)\,d\beta$$

We can regularize the integrand as usual with a Hilbert transform:

$$w(\zeta(\alpha)^+) = \frac{1}{2}\mu(\alpha) + \frac{1}{2\pi i}\int_0^{2\pi}\left[\frac{\zeta'(\beta)}{2}\cot\left(\frac{\zeta(\alpha)-\zeta(\beta)}{2}\right) - \frac{1}{2}\cot\left(\frac{\alpha-\beta}{2}\right)\right]\mu(\beta)\,d\beta$$

$$+ \underbrace{\frac{1}{2\pi i}\fint_0^{2\pi}\frac{1}{2}\cot\left(\frac{\alpha-\beta}{2}\right)\mu(\beta)\,d\beta}_{-\frac{i}{2}H\mu(\alpha)}$$

We solve for μ by setting $\mathrm{Re}\{w(\zeta(\alpha)^+)\} = \Phi$

value of ϕ on the surface

$$\frac{1}{2}\mu(\alpha) + \frac{1}{2\pi}\int_0^{2\pi} K(\alpha,\beta)\mu(\beta)\,d\beta = \Phi(\alpha)$$

$$K(\alpha,\beta) = \begin{cases} \mathrm{Im}\left\{\dfrac{\zeta'(\beta)}{2}\cot\left(\dfrac{\zeta(\alpha)-\zeta(\beta)}{2}\right)\right\} & \alpha \neq \beta \\[2ex] -\mathrm{Im}\left\{\dfrac{\zeta''(\alpha)}{2\zeta'(\alpha)}\right\} & \alpha = \beta \end{cases}$$

Next we want to compute the velocity.

Cauchy-Riemann: $\phi_y = -\psi_x$

$$u - iv = \phi_x - i\phi_y = \phi_x + i\psi_x = W_z$$

$$= \frac{1}{2\pi i}\int_0^{2\pi}\left[\frac{\partial}{\partial z}\,\frac{\zeta'(\alpha)}{2}\cot\left(\frac{z-\zeta(\alpha)}{2}\right)\right]\mu(\alpha)\,d\alpha$$

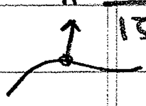
$$= \frac{1}{2\pi i}\int_0^{2\pi}\left[-\frac{\partial}{\partial\alpha}\,\frac{1}{2}\cot\left(\frac{z-\zeta(\alpha)}{2}\right)\right]\mu(\alpha)\,d\alpha$$

$$= \frac{1}{2\pi i}\int_0^{2\pi}\frac{1}{2}\cot\left(\frac{z-\zeta(\alpha)}{2}\right)\underbrace{\mu'(\alpha)}_{\gamma(\alpha)}\,d\alpha$$

$\gamma(\alpha) \leftarrow$ vortex sheet strength

we get $G\Phi = \underbrace{\sqrt{1+\eta_x^2}}_{|\zeta'|}\,\dfrac{\partial\phi}{\partial n}$ from

$\hat{n} = \dfrac{(-\eta_x, 1)}{|\zeta'|}$



$$(u,v)\cdot\frac{(-\eta_x, 1)}{|\zeta'|} = -\operatorname{Im}\left\{\frac{(u-iv)(1+i\eta_x)}{|\zeta'|}\right\}$$

$$|\zeta'(\alpha)|\,\frac{\partial\phi}{\partial n}(z,\alpha) = -\operatorname{Im}\left\{\frac{1}{2\pi i}\int_0^{2\pi}\frac{\zeta'(\alpha)}{2}\cot\left(\frac{z-\zeta(\beta)}{2}\right)\gamma(\beta)\,d\beta\right\}$$

Taking the limit as $z \to \zeta(\alpha)^+$, we get

$$\underbrace{|\zeta'(\alpha)|\,\frac{\partial\phi}{\partial n}(\zeta(\alpha)^+)}_{G\Phi(\alpha)} = -\operatorname{Im}\left\{\frac{1}{2}\gamma(\alpha) + \frac{1}{2\pi i}\int_0^{2\pi}\frac{\zeta'(\alpha)}{2}\cot\left(\frac{\zeta(\alpha)-\zeta(\beta)}{2}\right)\gamma(\beta)\,d\beta\right\}$$

Finally, we regularize the integrand (and use $-\mathrm{Im}(A) = \mathrm{Re}\{iA\}$)

$$\mathcal{G}\Phi(\alpha) = \mathrm{Re}\left\{ \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\zeta'(\alpha)}{2}\cot\left(\frac{\zeta(\alpha)-\zeta(\beta)}{2}\right) - \frac{1}{2}\cot\left(\frac{\alpha-\beta}{2}\right)\right]\gamma(\beta)\,d\beta \right.$$
$$\left. + \frac{1}{2\pi}⨍ \frac{1}{2}\cot\left(\frac{\alpha-\beta}{2}\right)\gamma(\beta)\,d\beta \right\}$$

$$\mathcal{G}\Phi(\alpha) = \frac{1}{2}H\gamma(\alpha) + \frac{1}{2\pi}\int_0^{2\pi} G(\alpha,\beta)\,\gamma(\beta)\,d\beta$$

$$G(\alpha,\beta) = \begin{cases} \mathrm{Re}\left\{ \frac{\zeta'(\alpha)}{2}\cot\left(\frac{\zeta(\alpha)-\zeta(\beta)}{2}\right) - \frac{1}{2}\cot\left(\frac{\alpha-\beta}{2}\right)\right\} & \alpha \neq \beta \\ \mathrm{Re}\left\{ \frac{\zeta''(\alpha)}{2\zeta'(\alpha)} \right\} & \alpha = \beta \end{cases}$$

To compute the pressure, we go back to Bernoulli:

$$\phi_t + \frac{1}{2}\|\nabla\phi\|^2 + \frac{P}{\rho} + gy = C(t)$$

In the traveling case, we had $C(t) = 0$ and $\phi_t = -c\phi_x$

so

$$P(z) = \rho\left(c\phi_x - \frac{1}{2}\phi_x^2 - \frac{1}{2}\phi_y^2 - gy\right)$$

where we use the formula $u - iv = \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{2}\cot\left(\frac{z-\zeta(\alpha)}{2}\right)\gamma(\alpha)\,d\alpha$

to evaluate $\phi_x = u$, $\phi_y = v$ at each point z of the mesh.