method at \( t_{n+1} \).

discretize space first, leaving time continuous

e.g. 2nd order space: \( \frac{\partial u}{\partial t} = D_x^+ D_x^- u \),
\( \frac{\partial u}{\partial t} = B u \),
\( B = D_x^+ D_x^- \)

Then discretize this ODE in time using a scheme from 228A.

e.g. trapezoidal rule: \[
\begin{align*}
U^{n+1} &= U^n + \frac{\Delta t}{2} \left[ f(U^n) + f(U^{n+1}) \right]
\end{align*}
\]

in our case, \( f(u) = Bu \) is linear, but you can also use this approach for nonlinear PDE's.

The trapezoidal rule is a Runge-Kutta method with Butcher array

\[
\begin{array}{c|cccc}
  c_i & A_i \\
  \hline
  0 & 0 & 0 & 0 & 0 \\
  1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array}
\]

general Runge-Kutta method \((s\text{-stage})\), solving the ODE \( U' = f(U) \)

\[
\begin{align*}
l_i &= f \left( U^n + k \sum_{j=1}^{s} a_{ij} l_j \right), \quad 1 \leq i \leq s \\
U^{n+1} &= U^n + k \sum_{j=1}^{s} b_i l_j \\
\end{align*}
\]
the \( l_i \) are called stage derivatives

truncation error: \( E_n = \frac{k}{2} \left[ l(U(t_n+k)) - l(U(t_n)) - k \sum_{j=1}^{s} b_j \frac{d}{dt} l_j(k) \right] \)

\( l(t) \) is exact solution; scheme initialized with \( U(t_n) \)
then we Taylor expand the solution and the scheme

\[ u(t_n + k) = u(t_n) + k u'(t_n) + \ldots + \frac{k^{p+1}}{(p+1)!} u^{(p+1)}(t_n) + O(k^{p+2}) \]

\[ l_i^j(k) = f(u + k \sum_{j=1}^{i} a_{ij} l_j^i(k)) \]

\[ = l_i^j(0) + k l_i^j(1) + \ldots + \frac{k^p}{p!} l_i^j(p) + O(k^{p+1}) \]

result:

\[ T^n = \frac{1}{k} \left[ (u' - \sum_j b_j l_j^i) k + \left( \frac{1}{2} u'' - \sum_j b_j l_j^i \right) k^2 \right] \]

\[ + \ldots + \left( \frac{1}{(p+1)!} u^{(p+1)} - \frac{1}{p!} \sum_j b_j^{(p)} \right) k^{p+1} \]

+ \( O(k^{p+1}) \)

tool for evaluating these terms: labeled trees

formulas:

\[ u^{(q)}(t_n) = \sum_{\phi \in T_q} \alpha(\phi) F(\phi)(u(t_n)) \]

\[ l_j^{(q-1)}(0) = \frac{1}{q} \sum_{\phi \in T_q} \alpha(\phi) \chi(\phi) \Phi_j(\phi) F(\phi)(u(t_n)) \]

\( T_q = \) set of labeled trees of order \( q \)

\( \alpha(\phi) = \) \# of ways of labeling nodes (an integer)

\( \chi(\phi) = \) product of subtree orders (another integer)

\( \Phi_j(\phi) = \) "tree product" of Butcher array entries

\( F(\phi)(u) = \) elementary differential
Examples: \( F(\mathbf{a}, \mathbf{b}) = D^3 f(u) \left( \frac{\partial^2 f(u)}{\partial (f(u))}, f(u), Df(u)(f(u)) \right) \)

\[ \Phi_j(\mathbf{k}, \mathbf{l}) = \sum_{\mathbf{m}, \mathbf{n}, \mathbf{p}} \alpha_{jkmnlp} \alpha_{j} \]

Where do elementary differentials come from?

\( u' = f(u) \)

\( u'' = Df(u)u' = Df(u)f(u) \)

\( u''' = D^2 f(u) (f(u), f(u)) + Df(u) Df(u) f(u) \)

We've been doing this already with our PDE's:

\( u_t = U_{xx} \)

\( u_{tt} = u_{txx} = u_{xxxx} \)

\( u_{ttt} = u_{xxxxx} \)

The term corresponding to \( \sqrt{ } \) is zero since the PDE is linear:

\( f(u) = U_{xx} \)

\( Df(u)(v) = \frac{d}{dv} f(u + v) = V_{xx} \)

\( D^2 f(u)(v, w) = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} Df(u+\varepsilon w)(v) = \frac{d}{d\varepsilon} V_{xx} = 0 \)
in the formula for \( T^n \) above, \( u', u'', \) etc. refer to the exact solution of the ODE that we get after discretizing space. To compute the truncation error of the PDE, we have to replace these with \( u_t, u_{tt}, \) etc. for the PDE. This just introduces terms in \( T^n \) of the form

\[
\frac{1}{q!} \left( \frac{\partial^q u_{PDE}}{\partial t^q} - u_{ODE} \right) k^q
\]

We use trees to write \( \frac{\partial^q u_{PDE}(x, t_n)}{\partial x^q} = \sum_{\phi \in T_q} \alpha(\phi) F_{PDE}(\phi)(u(x, t_n)) \)

Final result: (drop \( n \) from \( T^n \))

\[
T = \frac{1}{k} \left[ \left( \frac{\partial u_{PDE}}{\partial x} - \sum_j b_i^j(x) \right) k + \ldots + \left( \frac{1}{(p+1)!} \frac{\partial^{p+1} u_{PDE}}{\partial x^{p+1}} - \sum_j b_i^j(x) \right) k^{p+1} \right]
\]

\[
= \frac{1}{k} \left[ T^{(1)} k + \frac{T^{(2)} k^2}{2} + \ldots + \frac{T^{(p+1)} k^{p+1}}{(p+1)!} \right] + O(k^{p+1})
\]

\[
T^{(q)} = T^{(q)}_{\text{space}} + T^{(q)}_{\text{time}}
\]

\[
T^{(q)}_{\text{space}} = \sum_{\phi \in T_q} \alpha(\phi) \left[ F_{PDE}(\phi)(u) - F_{ODE}(\phi)(u) \right]
\]

\[
T^{(q)}_{\text{time}} = \sum_{\phi \in T_q} \alpha(\phi) \left[ 1 - \delta(\phi) \sum_{j=1}^{s} b_i^j \Phi_i^j(\phi) \right] F_{ODE}(\phi)(u)
\]

here we use the Runge-Kutta order conditions

\[
\sum_{j=1}^{s} b_i^j \Phi_i^j(\phi) = \frac{1}{\delta(\phi)} \quad \phi \in T_q \quad 1 \leq q \leq p
\]
Let's bring this down to Earth by returning to the original example

\[ \text{PDE: } U_t = U_{xx} \]

\[ \text{ODE: } U_t = Bu, \quad B = D_x^2D_x^{-1} \]

\[ \text{Scheme: } \begin{array}{c|c|c}
\hline
0 & 0 & 0 \\
\hline
1 & -\frac{1}{\Delta t} & \frac{1}{\Delta t^2} \\
\hline
\end{array} \quad \text{trap. rule} \quad \text{2nd order, so } p=2 \]

\[ \tau = \frac{1}{k} \left[ \tau_1^{(1)} + \tau_2^{(2)} \frac{k^2}{2} + \tau_3^{(3)} \frac{k^3}{6} \right] + O(k^3) \]

\[ q=1, \quad \tau_1^{(1)} = \{ 0 \} \quad \text{(only one term of order 1)} \]

\[ \tau_1^{(1)} = \alpha(\cdot) \left[ \frac{F_{\text{PDE}}(u) - F_{\text{ODE}}(u)}{\frac{1}{\Delta x^2} U_{xx} - Bu} \right] \]

\[ = U_{xx} - \left[ U_{xx} + \frac{k^2}{12} U_{xxxx} + \cdots \right] = -\frac{k^2}{12} U_{xxxx} + \cdots \]

\[ \tau_1^{(1)} = 0 \]

\[ q=2, \quad \tau_2 = \{ 0 \} \]

\[ \tau_2^{(2)} = \alpha(\cdot)^2 \left[ \frac{P_{\text{PDE}}(u)}{U_{xxxx}} - \frac{P_{\text{ODE}}(u)}{Bu} \right] \]

\[ B^2u = B \left[ U_{xx} + \frac{k^2}{12} U_{xxxx} + \cdots \right] = U_{xxxx} + \frac{k^2}{6} U_{6x} + \cdots \]

\[ \tau_2^{(2)} = -\frac{k^2}{6} U_{6x} + \cdots \quad \tau_2^{(2)} = 0 \]
\( q = 3 \quad T_3 = \{ \mathcal{V}, \mathcal{S}, \mathcal{G} \} \)

Both operators \( f_{\operatorname{PDE}} \) and \( f_{\operatorname{ODE}} \) have \( F(\mathcal{V})(u) = 0 \)

\[
T^{(3)}_{\text{Space}} = \frac{\alpha(3)}{1} \left[ Df(u) \left( Df(u) \left( f(u) \right) \right) - Df(u) \left( Df(u) \left( f(u) \right) \right) \right] \frac{u_{6x}}{B^3 u} \frac{1}{\alpha(3)} \left[ \frac{1}{6} \left( \sum_{j=1}^{2} \mathbf{E}_j \right) \right] F_{\operatorname{ODE}}(u) \frac{u_{8x}}{B^3 u} \frac{1}{\alpha(3)} \left[ \frac{1}{6} \left( \sum_{j=1}^{2} \mathbf{E}_j \right) \right]

\[
B^3 u = B \left[ u_{6x} + \frac{h^2}{6} u_{6x} + \cdots \right] = u_{6x} + \frac{h^2}{4} u_{8x} + \cdots
\]

\[
T^{(3)}_{\text{Space}} = -\frac{h^2}{4} u_{8x} + \cdots
\]

\[
T^{(3)}_{\text{Time}} = \frac{\alpha(3)}{1} \left[ 1 - \frac{\gamma(3)}{6} \sum_{j=1}^{2} \mathbf{E}_j \right] \frac{u_{6x}}{B^3 u} \frac{1}{\alpha(3)} \left[ \frac{1}{6} \left( \sum_{j=1}^{2} \mathbf{E}_j \right) \right] F_{\operatorname{ODE}}(u) \frac{u_{8x}}{B^3 u} \frac{1}{\alpha(3)} \left[ \frac{1}{6} \left( \sum_{j=1}^{2} \mathbf{E}_j \right) \right]
\]

\[
= \left( 1 - \frac{3}{2} \right) B^3 u = -\frac{1}{2} \left[ u_{6x} + \frac{h^2}{4} u_{8x} + \cdots \right]
\]

Put it all together:

\[
T = -\frac{h^2}{12} u_{xxxx} - \frac{h^2}{12} u_{6x} - \frac{h^2}{24} u_{8x} - \frac{h^2}{12} u_{6x} - \frac{h^2}{48} u_{8x} - \frac{h^2}{16} u_{8x} + O(h^4 + k^3)
\]
this formula for \( C \) is different than the one in the notes on page 65 because

1. in the notes we expanded about \( (x_j, \tau_n + \frac{k}{2}) \)
   while here we expanded about \( (x_j, \tau_n) \)

2. for implicit schemes, when we say "\( C \) is what's left over when you plug the exact solution into the scheme," we really mean "initialize the scheme with the exact solution at \( \tau_n \), then compare the exact solution at \( \tau_{n+1} \) to the result of the scheme." I did it the first way (i.e., incorrectly) in the notes.
Last time: method of lines (truncation error, Runge-Kutta order conditions)

\[ \text{stability} \]

ODE after discretization in space: \( U_t = BU \)

\( B \) a finite difference operator

recall linear stability analysis for Runge-Kutta:

\[ \dot{y} = \lambda y, \quad \lambda \in \mathbb{C} \]

\[ l_i = \lambda \left( y_n + k \sum_{j=1}^{s} a_{ij} l_j \right) \]

\[ (I - k\lambda A)l = \lambda y_n e, \quad e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad l = \begin{pmatrix} l_i \\ l_j \end{pmatrix} \in \mathbb{C}^s \]

\[ y_{n+1} = y_n + k \beta_l = \left[ I + k\lambda \beta_l (I - k\lambda A)^{-1} \right] y_n \]

\[ y_n = R(k\lambda)^n y_0 \]

now consider one Runge-Kutta step of \( U_t = BU \)

\[ l_i = B \left( u^n + k \sum_{j=1}^{s} a_{ij} l_j \right) \]

we can reduce to the previous case by diagonalizing \( B \): \( \hat{B} \)

\[ \hat{B} \rightarrow \text{diag}(\hat{B}) \]

\[ Z \hat{B} Z^{-1} \]

\[ \hat{l}_i = \hat{Z} \hat{B}^{-1} \left( \hat{Z} u^n + k \sum_{j=1}^{s} a_{ij} \hat{Z} l_j \right) \]

\[ \hat{Z} \hat{l}_i(\hat{Z}) = \hat{Z} \hat{B}^{-1} \left( \hat{Z} u^n + k \sum_{j=1}^{s} a_{ij} \hat{Z} l_j(\hat{Z}) \right) \]
with the correspondence \( y_n \leftrightarrow \hat{u}^n(\xi), \quad \lambda \leftrightarrow G(\xi) \) we obtain

\[
\hat{u}^n(\xi) = R( \lambda G(\xi) ) \hat{u}^0(\xi)
\]

or

\[
u^n_j = [R( \lambda B )]^n u^0_j \]

where \( R( \lambda B ) = Z^{-1} R( \lambda G(\xi) ) Z \) - usual functional calculus

**Conclusion:** the method of lines is **stable** if \( \exists \Sigma, K \) s.t.

\[
||R(\lambda B(\xi))|| \leq K \quad \text{for} \quad 0 \leq \Re \lambda \leq T
\]

(In general, \( B \) depends on \( \xi \), in that case, we write \( G(\xi, \lambda) \))

A sufficient condition:

\[
|R(\lambda G(\xi, \lambda))| \leq 1 + C K \quad \text{for} \quad -\pi \leq \xi \leq \pi
\]

**Example:** Crank-Nicolson

**Space:** \(Bu_j = D_x^+ D_x^- u_j = \frac{1}{h} \left[ u_{j+1} - 2u_j + u_{j-1} \right]\)

\(G(\xi) = \frac{1}{h} \left[ e^{i \xi} - 1 + e^{-i \xi}\right] = \frac{2}{h^2} (1 - \cos \xi) = -\frac{4}{h^2} \sin^2 \left(\frac{\xi}{2}\right)\)

**Time:** \(y' = \lambda y\), \(y^{n+1} = y^n + \frac{h}{2} \left[ \lambda y^n + \lambda y^{n+1} \right]\)

\[\left(1 - \frac{h \lambda}{2}\right)y^{n+1} = \left(1 + \frac{h \lambda}{2}\right)y^n \quad y^{n+1} = \frac{1 + \frac{h \lambda}{2}}{1 - \frac{h \lambda}{2}} y^n\]

**Combined:** \[|R(\lambda G(\xi))| = \left|\frac{1 - 2v \sin^2 \left(\frac{\xi}{2}\right)}{1 + 2v \sin^2 \left(\frac{\xi}{2}\right)}\right| \leq 1\] (unconditionally stable)
RAI = \{ z \in \mathbb{R} : |e^{z} - 1| \leq 1 \}

\downarrow

since the trapezoidal rule is A-stable (the region of absolute stability contains the left half-plane) we will have

\[ |R(k \ell(z))| \leq 1 \]

provided that \( \text{Re}(G(z)) \leq 0 \). This is sometimes too much to hope for, e.g. when solving

\[ u_t = u_{xx} + u \]

the exact solution grows, so we need \( \| R(k \ell) \| > 1 \) so the numerical solution can grow. If the Runge-Kutta method is A-stable, then

\[ R(z) = \frac{p(z)}{Q(z)} = \frac{p_0 + p_1 z + \cdots + p_{\beta} z^\beta}{q_0 + q_1 z + \cdots + q_{\beta} z^\beta} \]

with \( \alpha \leq \beta \leq \delta \) and \( |q_{\beta}| \leq |q_{\beta-1}| \). (\( \beta = 0 \) is allowed) otherwise \( |R(z)| \) would blow up \( (\alpha > \beta) \) or approach \( \frac{1}{q_{\beta}} > 1 \)
as \( z \to \infty \). Moreover, all the poles of \( R(z) \) lie in the right half plane. We conclude that

\[ R'(z) = \frac{Q(z)P'(z) - P(z)Q'(z)}{Q(z)^2} = \frac{\tilde{P}}{Q^2} \quad \text{deg} \tilde{P} < \text{deg} Q = 2\beta \]
is bounded in \( C_{\xi} = \{ z : \text{Re} z \leq \xi \} \) for some \( \xi > 0 \). Poles

This leads to a simple stability condition for the method of lines.
Theorem: If \( \exists \epsilon_1, \epsilon_2 \) s.t. \( \text{Re} \{ G(\xi, k) \} \leq C_1 \) for \( 0 < k < \epsilon_1 \) and the time-stepping scheme is A-stable, then the method of lines is stable.

Proof: Let \( \epsilon_2 > 0 \) be constants so that \( |R'(\xi)| \leq C_2 \) for \( \xi \in \epsilon_2 \).

Let \( \epsilon = \min(\epsilon_1, \frac{\epsilon_2}{C_1}) \). Write \( kg(\xi, k) = a + ib \).

Then for \( 0 < k < \epsilon \) we have

\[
\begin{align*}
    a &= k \text{Re} \{ G(\xi, k) \} < \frac{\epsilon_2}{C_1} = \epsilon_2 \\
    R(a + ib) &= R(ib) + \int_0^1 \frac{d}{d\theta} R(ib + \theta a) a d\theta \\
    |R(a + ib)| &\leq |R(ib)| + C_2 a \leq 1 + C_2 C_1 k \\
    |R(G(\xi, k))| &\leq 1 + Ck, \quad \text{Scheme is stable.}
\end{align*}
\]

Note: we assumed \( a > 0 \). If \( a < 0 \), we know \( |R(a + ib)| \leq 1 \) and the scheme is A-stable.

Example: \( u_t = u_{xx} + u_x + u \)

\[
    u_t = Bu \quad B_u = \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + \frac{U_{j+1} - U_{j-1}}{h} + U_j
\]

\[
    G(\xi, k) = \frac{1}{h} \left[ -4 \sin^2 \frac{\xi}{2} \right] + \frac{i}{h} \left[ 2i \sin \frac{\xi}{2} \right] + 1
\]

Still need to specify a refinement path, \( h(k) \).
Recap: \[ U_t = -aU_x \]

Each Fourier mode travels with speed \( a \):

\[ u(x,0) = e^{i \omega x} \Rightarrow u(x,t) = e^{i(x-at)} \]

\[ u(x,t+k) = e^{-i a \omega \frac{t}{k}} u(x,t) \]

**Numerical solution**

\[ u_j^n = \mathcal{F}^i u \Rightarrow u_{j+1}^{n+1} = \mathcal{F}(\mathcal{F}^i u) \]

\[ \mathcal{F}(\mathcal{F}^i u) = p(\tau) e^{-i \omega \frac{\tau}{k}} \]

**Numerical solution is exact if** \( p(1) = 1, \omega(1) = a \)

**Theorem:** The scheme is order \( r \) if

\[ p(\tau) = 1 + O(\tau^r) \]

\[ \frac{\omega(\tau)}{a} = 1 + O(\tau^r) \]

**Proof:** These conditions imply that \( \mathcal{F}(\mathcal{F}^i u) = e^{-i \omega \frac{\tau}{k}} = O(\tau^{r+1}) \)

(just note that \( \omega(\tau)\frac{\tau}{k} = \omega \tau + O(\tau^{r+1}) \))

(gives extra order)

Back to original def. of \( \mathcal{F} \)-transform:

\[ \mathcal{F}^n = \frac{1}{k} \left[ \sum_{i=1}^{N} u_i(x_i, nh_k) - (B \sum_{i=1}^{N} u_i(x_i, nh_k)) \right] \]

\[ \mathcal{F}^n = \frac{1}{k} \left[ e^{-i \omega \frac{\tau}{k}} - \mathcal{F}(\mathcal{F}^i u) \right] U(\cdot, nh_k)(\mathcal{F}) \]

\[ \mathcal{F}^n = \frac{1}{k} \left[ (i \omega \frac{\tau}{k})^r e^{-i \omega \frac{\tau}{k}} - \mathcal{F}(\mathcal{F}^i u) \right] U(\cdot, nh_k)(\mathcal{F}) \]

\[ = A(\mathcal{F}) \left[ (i \omega \frac{\tau}{k})^r e^{-i \omega \frac{\tau}{k}} - \mathcal{F}(\mathcal{F}^i u) \right] U(\cdot, nh_k)(\mathcal{F}) \]

\[ = A(\mathcal{F}) \left[ (i \omega \frac{\tau}{k})^r e^{-i \omega \frac{\tau}{k}} - \mathcal{F}(\mathcal{F}^i u) \right] U(\cdot, nh_k)(\mathcal{F}) \]

\[ = A(\mathcal{F}) \left[ (i \omega \frac{\tau}{k})^r e^{-i \omega \frac{\tau}{k}} - \mathcal{F}(\mathcal{F}^i u) \right] U(\cdot, nh_k)(\mathcal{F}) \]

\[ = A(\mathcal{F}) \left[ (i \omega \frac{\tau}{k})^r e^{-i \omega \frac{\tau}{k}} - \mathcal{F}(\mathcal{F}^i u) \right] U(\cdot, nh_k)(\mathcal{F}) \]
\[ \frac{1}{\sqrt{2\pi}} \frac{1}{\text{a}} \int_{-\infty}^{\infty} |C_n(\xi)|^2 d\xi \]

\[ \frac{1}{\sqrt{m}} \sum_{m=0}^{\infty} |m|^2 \left| C_n(\frac{m}{\text{a}}) \right|^2 \]

\[ \Rightarrow \| \xi n \|_{L_2} \leq (M \text{nu}^r) \| u^{(n+1)}(x) \|_{L_2(\mathbb{R})} + O(h^m) \]

In the last step, we use the fact that the \( \xi \)-transform is closely related to the Fourier transform.

Given \( v(x) \), let \( \tilde{v}(\lambda) = \int_{-\infty}^{\infty} v(x) e^{i\lambda x} dx \)

Poisson summation formula:

\[ \sum_{j} v(jh) e^{i j \frac{\pi}{h}} = \sum_{n} \frac{1}{h} \tilde{v} \left( \frac{\pi + 2\pi n}{h} \right) \]

\[ \tilde{v}(\frac{\pi}{h}) = \frac{1}{h} \tilde{v}(\frac{\pi}{h}) + \sum_{n \neq 0} \frac{1}{h} \tilde{v} \left( \frac{\pi + 2\pi n}{h} \right) \]

\( \xi \)-transform of sampled Fourier transformalising error

Fourier transform of derivatives:

\[ (i \lambda)^m \tilde{v}(\lambda) = \int v(x) (i \lambda)^m e^{-i\lambda x} dx = \int v^{(m)}(x) e^{-i\lambda x} dx \]

\[ \left( i \frac{\pi}{h} \right)^m \tilde{v}(\frac{\pi}{h}) = \frac{1}{h} \tilde{v}^{(m)}(\frac{\pi}{h}) + \sum_{n \neq 0} \frac{1}{h} \tilde{v}^{(m)} \left( \frac{\pi + 2\pi n}{h} \right) \]
Thus, \[
\left( \frac{\hbar}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{h} \sqrt{m} \left( \frac{\xi}{h} \right) \right|^2 d\xi \right)^{1/2} \leq \left( \frac{\hbar}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{h} \sqrt{m} \left( \frac{\xi}{h} \right) \right|^2 d\xi \right)^{1/2} + \left( \frac{\hbar}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{h} \sqrt{m} \left( \frac{\xi + 2\pi n}{h} \right) \right|^2 d\xi \right)^{1/2}.
\]

2nd term bounded using \( \left| A^2 \sqrt{m^2}(x) \right| \leq C_{m^2} = \int |V(m^2)(x)| dx \)

\[
\sum_{n \neq 0} \frac{1}{h} \sqrt{m} \left( \frac{\xi + 2\pi n}{h} \right) \leq \frac{2}{\hbar} \sum_{n=1}^{\infty} \frac{C_{m^2}}{(2\pi n)^2} \left| \xi + 2\pi n \right| \geq 2\pi n - |\xi|
\]

\[
= \frac{2\hbar C_{m^2}}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\hbar C_{m^2}}{4}
\]

\[
\therefore \text{2nd term } \leq \frac{\hbar^{3/2} C_{m^2}}{4}
\]

1st term computed using change of variables

\[
\frac{\hbar}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{h} \sqrt{m} \left( \frac{\xi}{h} \right) \right|^2 d\xi \quad \xi = \eta h
\]

\[
d\xi = h \, d\eta
\]

\[
= \frac{\hbar^2}{2\pi} \int_{-\eta h}^{\eta h} \left| \frac{1}{h} \sqrt{m} (\eta) \right|^2 d\eta
\]

\[
\approx \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \sqrt{m}(\eta) \right|^2 d\eta = \int_{-\infty}^{\infty} |V(m)(x)|^2 dx
\]

\[
\text{error is } O(h^3) \text{ since } \sqrt{m}(\lambda) \leq \frac{C_{m^2}}{\lambda^2}
\]

\[
\text{error } \leq \frac{2}{2\pi} \int_{-\eta h}^{\eta h} \frac{C_{m^2}}{\lambda^2} d\lambda = \frac{C_{m^2}}{\pi} \left( \frac{1}{\eta^2} \right)^3 = \frac{C_{m^2} \hbar^3}{3\pi^4}
\]

\[
\text{Plancherel's theorem}
\]
(add to discussion on Fourier collocation/pseudo-spectral methods  
Page 147 of 228B notes)

**filtering.** High frequency modes are not likely to be accurate.  
(not enough grid pts to resolve them).

But they get amplified the most when we take a spectral derivative,  
\[ B = D \]

\[ G = i h^{-1} \delta \]

\[ -\pi \]

\[ \pi \]

By contrast, \( D_x^n \) suppresses these modes,  

\[ G = i h^{-1} \sin \delta \]

but it only 2nd order accurate.  
\[ (i h^{-1} \delta - i h^{-1} \sin \delta) = O(h^3) \]

\[ +1 \]

A filtered version of \( D \) gives the best of both worlds:  

\[ G(\delta) = \frac{i h^{-1} \delta q(\delta)}{\text{filter}} \]

Common choices:  
\[ \frac{2}{3} \text{ rule: } q(\delta) = \begin{cases} 1 & |\delta| \leq \frac{2}{3} \pi \\ 0 & \text{o.w.} \end{cases} \]

36 rule:  
\[ q(\delta) = e^{-36(\frac{\delta}{\pi})^{36}} \]  
(36th order method)

\[ \frac{2}{3} \text{ rule} \]

\[ \frac{36}{36} \text{ rule} \]

\[ D^n u_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} -h^{-2} \sin \frac{\pi \gamma}{h} e^{i \pi \gamma} u_j d\gamma \]
Hyperbolic conservation laws

\[ u_t + F(u)_x = 0 \]

\[ u \in \mathbb{R}^m, \quad u_i \text{ state variable (mass, momentum, energy...)} \]

\[ F = \text{flux function} \]

Example:

1. Traffic flow:
   \[ p_t + (pU(p))_x = 0 \]
   
   \[ p = \text{density of cars (flow per car length), \quad 0 \leq p \leq 1} \]
   
   \[ U(p) = \text{velocity} \quad (U(0) = \text{speed limit}, \quad U(1) = 0 \text{mp/h}) \]

   \[ F(p) = pU(p) = \text{flux of cars (cars/sec per pos)} \]

2. Compressible gas dynamics:

   \[ \frac{2}{\alpha} \left[ \frac{p \nu}{E} \right] + \frac{2}{\alpha} \left[ \frac{p \nu}{\nu^2 + p} \right] = 0 \]

   \[ p = \text{density}, \quad p = \text{pressure}, \quad \nu = \frac{k}{\gamma - 1}, \quad p(p) = kp^\gamma \]

   \[ \nu = \text{velocity}, \quad \rho \nu = \text{momentum}, \quad \text{equation of state: (ideal gas law)} \]

   \[ E = \text{energy} \]

3. Isentropic Euler (debye-nergy):

   \[ \frac{2}{\alpha} \left( \frac{p \nu}{E} \right) + \left( \frac{p \nu}{\nu^2 + kp^\gamma} \right)_x = 0 \]
4. Shallow water eqs.\[ \frac{\partial h}{\partial t} + (vh)x = 0 \]
\[ \frac{\partial v}{\partial t} + (v^2/2 + gh)x = 0 \]

- \( v \): horizontal velocity
- \( h \): height
- \( g \): gravity

5. Burgers' equation
\[ u_t + uu_x = 0 \]
\[ F(u)_x = uu_x \]

**Burgers' equation**

Induced limit of \( u_t + uu_x = 0 \) is called \( \text{Non-Skew} \)

\[ p(u_t + u - Du_x) = -\nabla p + \mu \Delta u \]
\[ \nabla u = 0 \]

**Continuity equation:** \[ p_t + (pu)_x = 0 \]

follow the material

\[ \alpha = a(\alpha), \quad \beta = \beta(\beta) \]

\[ 0 = \frac{d}{dt} \int_a^b p \, dx = \int_a^b p_t \, dx + p(b) \beta - p(a) \alpha \\
= \int_a^b p_t + (pu)_x \, dx \]

Similarly,

\[ p(a) - p(b) = \frac{d}{dt} \int_a^b pv \, dx \]  \hspace{1cm} \text{(momentum equation)}

\[ \Rightarrow \int_a^b p_t + (pv^2 - p) \alpha \, dx \]
Characteristics.

\[ U_t + F(u) u_x = 0 \implies U_t + \frac{F'(u)}{a(u)} u_x = 0 \]

Let \( x(t) \) evolve in time and note that

\[ \frac{d}{dt} U(x(t), t) = U_x \dot{x} + U_t = 0 \quad (U \text{ remains constant}) \]

provided \( \dot{x}(t) = a(u(x(t), t)) = a(u_0) \)

\[
\begin{align*}
\frac{dt}{da} &= (\frac{dx}{dt})^{-1} = \frac{1}{a} \\
\end{align*}
\]

Example: Burgers’ eqn,

\[ U_t + U u_x = 0 \quad a = U \]

\[ U(x, 0) = \sin x \quad u_0(x) = \sin x \]

\[ x(t) = x_0 + a(u_0) t \]

\[ U(x_0 + t \sin(x_0), t) = \sin x_0 \]

\( x = x_0 + t \sin(x_0) \implies x_0(x(t)) \) solve implicitly,

\[ U(x_0(t), t) = \sin x_0(x(t)) \]

Except solution (valid until characteristics cross after which you can solve for \( x_0(x(t), t) \) uniquely)
general case: \( U_t + G(u_x, u, x, t) = 0 \), \( G(p, z, x, t) \) given.

solution propagates along characteristic \( \{ \begin{align*} x(t) &= U_t(x(t), t) \\ p(t) &= U_x(x(t), t) \\ q(t) &= U_t(x(t), t) \\ z(t) &= U(x(t), t) \end{align*} \) satisfying the ODEs:

\[
\begin{align*}
\dot{x} &= \theta(x, p, z, x, t) \\
\dot{p} &= -\partial_x \theta - \partial_z \dot{p} \\
\dot{q} &= -\partial_t \theta - \partial_z \dot{z} \\
\dot{z} &= p \dot{z} + q
\end{align*}
\]

duration
\[
\begin{align*}
\tau &= u_{xx} \dot{x} + u_x \dot{t} \\
\tilde{q} &= u_{tx} \dot{x} + u_t \dot{t}
\end{align*}
\]

use \( \Theta \) to eliminate 2nd derivatives:

\[
\begin{align*}
U_{tx} + \partial_x \theta \ u_{xx} + \partial_z \theta \ u_x + \partial_t \theta &= 0
\end{align*}
\]

define \( \dot{x} = \partial_x p \)

\[
\begin{align*}
U_{tt} + \partial_x \theta \ u_{tt} + \partial_z \theta \ u_t + \partial_t \theta &= 0
\end{align*}
\]

finally, \( \dot{z} = u_x \dot{x} + u_t = p \dot{x} + q \)
no office hours next week.

Last time: \( U_t + F(U)_x = 0 \)

Solutions are constant along characteristics:

\[
\begin{align*}
\frac{dx}{dt} &= a(U) = F'(U) \\
\frac{du}{dt} &= -F(U)
\end{align*}
\]

What should you do when characteristics cross?

Example: \( U_t + UU_x = 0 \), \( u(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases} \)

\[ s(t) \]

Shock:

To the left of the shock, \( u(x, t) = 1 \)

To the right, \( u(x, t) = 0 \)

The PDE is satisfied in each region, but the solution is discontinuous across the shock.

To make sense of discontinuous solutions, we need a more general definition:

**Weak solutions, version 1:**

\[
\frac{\partial}{\partial t} \int_a^b u(x,t) \, dx = F(u(a,t)) - F(u(b,t))
\]

This holds for smooth solutions since \( \int_a^b u_t \, dx = \int_a^b F'(u) \, du = F(u(b)) - F(u(a)) \)

But it also makes sense if there are shocks inside \([a, b]\).

**Shock speed:** let \( u_L = u(S(t^-), t^-) \), \( u_R = u(S(t^+), t^+) \)

\[ u_L = u(a, t^-) \quad u_R = u(b, t^+) \]

Suppose \([a, b]\) contains only one shock for \( t < t^+ \)

\[ u = u(a, t^-) \quad u = u(b, t^+) \]

\( \frac{d}{dt} \int_a^b u(x,t) \, dx = 0 \)
\[
\frac{\partial}{\partial t} \int_a^b u(x,t) \, dx = \frac{\partial}{\partial t} \left[ \int_a^s u \, dx + \int_s^b u \, dx \right]
\]

\[
F(u_R) - F(u_L) + u_L \frac{d}{dt} \left[ \int_a^s u \, dx \right] + F(u_R) - F(u_L) - u_R \frac{d}{dt} \left[ \int_a^b u \, dx \right] = F(u_R) - F(u_L)
\]

\[
\dot{S} = \frac{F(u_R) - F(u_L)}{u_R - u_L}
\]

Rankine-Hugoniot condition

\[
\dot{S} = \frac{[F(u)]}{[u]}
\]

(for system, can't divide by \([u]\) since it's a vector. need \((u_R - u_L) \dot{S} = F(u_R) - F(u_L))

Weak solution version 2: over any rect. \(x_1 < x < x_2\), \(t_1 < t < t_2\)

\[
\int_{x_1}^{x_2} u(x,t) \, dx + \int_{t_1}^{t_2} F(u(x,t)) \, dt = 0
\]

Weak solution version 3: consider \(\Delta A\) from \(\phi(x,t)\) with \(\phi \neq 0\)

\[
\int_0^\infty u \phi \, dt = u \phi \big|_0^\infty - \int_0^\infty u \phi \, dt = -u \phi \big|_{t=0} - \int_0^\infty u \phi \, dt
\]

\[
\int_0^\infty \int_0^\infty \left[ u \phi_t + F(u) \phi_x \right] \phi(x,t) \, dx \, dt = 0
\]

\[
\int_{-\infty}^\infty \int_{-\infty}^\infty u \phi_t + F(u) \phi_x \, dx \, dt + \int_{-\infty}^\infty u(x,0) \phi(x,0) \, dx = 0
\]

\(\forall \phi \in C^\infty_c(\mathbb{R}^\times \mathbb{R}^+)\)
\[ u + v = 0 \Rightarrow \]
\[ u + (\frac{u^2}{2})_x = 0 \]

\[ uu_x + v^2 u_x = 0 \Rightarrow (\frac{u^2}{2}) + (\frac{u^3}{3})_x = 0 \]

\[ s = \frac{F(u_O) - F(u)}{u_O - u} = \frac{\frac{1}{3} u^2}{u} = \frac{1}{2} (u_O + u) \]

\[ s = \frac{\left[ \frac{1}{2} u^2 \right]}{\left[ \frac{1}{3} u^2 \right]} = \frac{2}{3} \frac{u^3 - u_O^3}{u^3 - u_O^3} = \frac{2}{3} \frac{u_O^3 + u_O u_O + u_O^2}{u_O + u} \]

\[ s_2 - s_1 = \frac{1}{6} \frac{(u_O - u)^2}{u_O + u} \]

1. If considered physically relevant since we consider \( u^2 \) to be a density

2. Can be used as an entropy function (more later)
paraection waves.
characteristic cross when they move toward each other,
but they can also diverge, leaving a gap where the
solution is undetermined.

\[ u_t + uu_x = 0 \quad u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \]

Option 1: insert a shock
Option 2: rarefaction wave
Option 3: mixture of both

Option 1 yields a perfectly valid weak solution provided
\( s(t) \) satisfies the Rankine-Hugoniot condition.

Option 2 is a similarity solution:
\[ x \in \mathbb{R}, t > 0 \quad (u \text{ is constant along } \Gamma \text{ at } x = ct) \]
\[ u(x,t) = \phi \left( \frac{x}{t} \right) \]

To figure out \( \phi \), follow the characteristic (from the initial condition)
\[ \phi(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \]

\( \phi \) satisfies the PDE:
\[ u_t + uu_x = 0 \]

\[ \frac{\partial}{\partial t} \phi \left( \frac{x}{t} \right) + \phi \left( \frac{x}{t} \right) \frac{\partial}{\partial x} \phi \left( \frac{x}{t} \right) = 0 \]

Hence
\[ \phi \left( \frac{x}{t} \right) \left[ - \frac{x}{t^2} + \frac{1}{t} \phi \left( \frac{x}{t} \right) \right] = 0 \quad \Rightarrow \quad \phi \left( \frac{x}{t} \right) = 0 \text{ or } \phi \left( \frac{x}{t} \right) = \frac{1}{t} \]
to figure out $\phi$, plug into PDE

$$u_t + vu_x = 0$$

$$\frac{\partial}{\partial \xi} \phi \left( \frac{x}{t} \right) + \phi \left( \frac{x}{t} \right) \frac{2}{\xi} \phi \left( \frac{x}{t} \right) = 0$$

$$\phi' \left( \frac{x}{t} \right) \left[ -\frac{x}{t^2} + \frac{1}{t} \phi \left( \frac{x}{t} \right) \right] = 0$$

either $\phi' \left( \frac{x}{t} \right) = 0$ or $\phi \left( \frac{x}{t} \right) = \frac{x}{t}$

following characteristics (from the initial condition) to $t=1$

determines which case to use. In our case:

$t=1$

$$\phi(x) = \begin{cases} 
0 & x < 0 \\
\frac{x}{t} & 0 \leq x \leq 1 \\
1 & 1 < x 
\end{cases}$$

In option 3, we could imagine connecting $u=0$ on the left to $u=1$ on the right using any other continuous function at $t=1$

$$u(x, t) = \phi(x) = \begin{cases} 
0 & x > 0 \\
\frac{x}{t} & 0 \leq x \leq 1 \\
1 & x > 1 
\end{cases}$$

At any later time, the solution remains smooth. But if you go backward,
a shock will form before you reach $t=0$. 
following characteristics from \( t = 1 \), we find that

\[ u(x + \phi(x)(t-1), t) = \phi(x) \]

so

\[ u_x(x + \phi(x)(t-1), t) \left[ 1 + \phi'(x)(t-1) \right] = \phi'(x) \]

There exists \( \phi(x) > 1 \), then \( \phi = 0 \) before \( t \) reached \( 0 \) (\( x \) at \( \phi \))

and \( u_x(-\infty, t) = \infty \)

By the mean value theorem, if \( \phi(x) > x \) for some \( x \in (0, 1) \), there exists \( \phi(x) \) such that \( \phi'(x) = \frac{\phi(x) - 0}{x - 0} > 1 \)

so the only solution without shocks for \( 0 < t < 1 \) is the similarity solution (Riemann wave).

---

equal area rule:

formulas like \( u(x + \phi(x)t, t) = \phi(x) \)

make sense if you allow the solution to be multi-valued.

The Rankine-Hugoniot condition replaces the ocatom wave with a discontinuity that cuts off the same area to the right and left.
Entropy condition: want a condition to make weak solutions unique. Should be 1) equivalent to the vanishing viscosity solution 2) easy to check.

Idea: require that characteristics enter shocks (rather than emanate from them)

Lax entropy condition: (for scalar, convex conservation laws)

\[ f''(u) > 0 \text{ for all } u. \] (convexity)

\[ s'(t) > f'(u_L) \]

where \[ u_L = u(s(t)^-, t), u_R = u(s(t)^+, t) \]

in words: characteristics to the left travel faster than the shock, which travels faster the characteristics to the right.

This rules out evanescent but the redefinition fan when the initial condition has a discontinuity with \[ f'(u_L) < f'(u_R) \] (then must be no shock in that case)

Oleinik entropy condition: \[ U \] is the entropy solution (assuming \[ f''(u) > 0 \]) if \( E > 0 \) s.t. \( \forall a > 0, t > 0 \) and \( x \in \mathbb{R} \)

\[ \frac{u(x + at, t) - u(x, t)}{a} < \frac{E}{t} \]

This allows discontinuities \[ u_L \] but not \[ u_R \] at positive times.

(note \( f''(u) > 0 \) means increase \( u \) increases \( f'(u) \))
Entropy function

\[ \eta(u) = \text{entropy function} \]

\[ \Phi(u) = n \text{ flow} \]

When \( u \) is smooth, \( \eta(u)_t + \Phi(u)_x = 0 \)

(Entropy is conserved)

At an (admissible) shock, entropy (decreases) (sign convention opposite of physical entropy)

Entropy condition: \( \eta(u)_t + \Phi(u)_x \leq 0 \), integrated over a rectangle

\[ \int_{x_1}^{x_2} \eta(u(x,t)) \, dx \leq \int_{x_1}^{x_2} \eta(u(x, t_1)) \, dx \]

\[ \Delta x (\eta(u_1) - \eta(u_2)) + \Delta t (\Phi(u_1) - \Phi(u_2)) \leq 0 \]

Example: Burgers' \( \eta(u) = u^2 \)

\[ \Phi(u) = \frac{2}{3} u^3 \]

\[ s'(x) = \frac{u_L + u_R}{2} \]

\[ \mathcal{E} = u_R^2 - u_L^2 \]

\[ [\Phi] = \frac{2}{3} (u_R^3 - u_L^3) \]

Requirement: \( \frac{1}{2} (u_L + u_R)(u_R^2 - u_L^2) \geq \frac{2}{3} (u_R^3 - u_L^3) \)

\[ \frac{1}{6} (u_L - u_R)^3 \geq 0 \]

Need \( u_L > u_R \) to satisfy entropy condition.
conservation form: \( (equation: \ U_t + F(U)_x = 0) \)

\[
\frac{1}{h} \left( u_{i+1}^n - u_i^n \right) = -\frac{1}{h} \left[ \Gamma_{i+\frac{1}{2}}^n - \Gamma_{i-\frac{1}{2}}^n \right]
\]

\[ q = \# \ of \ grid \ pts \ on \ each \ side \ to \ use \ in \ numerical \ flux \ function \ \Gamma \]

\[ \Gamma_{i+\frac{1}{2}}^n = \Gamma(u_{i-q+1}^n, \ldots, u_{i+q}^n) \]

\[ \Gamma_{i-\frac{1}{2}}^n = \Gamma(u_{i-q}^n, \ldots, u_{i+q-1}^n) \]

\[ \frac{k}{h} = \nu \]

\text{examples: Lax-Friedrichs: } u_i^{n+1} = \frac{1}{2}(u_{i-1}^n + u_{i+1}^n) - \frac{k}{2h} \left[ F_{i+1}^n - F_{i-1}^n \right]

\[ \Gamma(u_0, u_1) = -\frac{1}{2\nu} (u_1 - u_0) + \frac{1}{2} \left[ F(u_1) + F(u_0) \right] \]

\text{check: } \Gamma(v, u) = F(u) \checkmark

\[-\frac{1}{h} \left[ \Gamma_{i+\frac{1}{2}}^n - \Gamma_{i-\frac{1}{2}}^n \right] = \frac{1}{2k} \left[ (u_{i+1}^n - u_i^n) - (u_i^n - u_{i-1}^n) \right] - \frac{1}{2h} \left[ (F_{i+1} + F_i) - (F_i - F_{i-1}) \right] \]

\[ = \frac{1}{k} \left[ \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{k}{2h} (F_{i+1}^n - F_{i-1}^n) - u_i^n \right] \]

\[ = \frac{1}{k} \left[ u_i^{n+1} - u_i^n \right] \checkmark \]
Lax-Wendroff

\[ U(x,t+h) = u + kut + \frac{k^2}{2} u_{tt} + \cdots \]

\[ u_t = -F(u)_x \]

\[ u_{tt} = - \left[ F'(u)u_t \right]_x = \frac{F'(u)F(u)_x}{a(u)} \]

\[ F(u)_x \approx D_0[F(u)] \]

\[ \left[ a(u)F(u)_x \right]_x \approx D^+ \left[ a_{i-\frac{1}{2}} D^- F(u) \right] \]

\[ = \frac{1}{h^2} \left[ a_{i+\frac{1}{2}} (F_{i+1} - F_i) - a_{i-\frac{1}{2}} (F_i - F_{i-1}) \right] \]

\[ a_{i+\frac{1}{2}} = \frac{1}{2} (a_i + a_{i+1}) \]

**Scheme:**

\[ u_{i+1}^{n+1} = u_i^n - \frac{\nu}{2} [F_{i+1} - F_{i-1}] + \frac{\nu^2}{2} \left[ a_{i+\frac{1}{2}} (F_{i+1} - F_i) - a_{i-\frac{1}{2}} (F_i - F_{i-1}) \right] \]

**Numerical Flux:**

\[ \Gamma_{i+\frac{1}{2}} = \frac{1}{2} (F_{i+1} + F_{i+1}) - \frac{\nu}{2} a_{i+\frac{1}{2}} (F_{i+1} - F_i) \]

**Check:**

\[ \Gamma(u, u) = \frac{1}{2} (F + F) - \frac{\nu}{2} a (F - F) = F(u) \checkmark \]

\[ \frac{k}{h} (\Gamma_{i+\frac{1}{2}} - \Gamma_{i-\frac{1}{2}}) = \frac{\nu}{2} (F_{i+1} - F_{i-1}) - \frac{\nu^2}{2} \left[ a_{i+\frac{1}{2}} (F_{i+1} - F_i) - a_{i-\frac{1}{2}} (F_i - F_{i-1}) \right] \]

\[ = -(u_{i+1}^{n+1} - u_i^n) \checkmark \]
last time: *3 ways to formulate entropy condition

* schemes in conservation form

\[ \begin{align*}
\text{equation:} & \quad U_t + F(U)_x = 0 \\
\text{scheme:} & \quad U_{j+1}^n = U_j^n - \frac{k}{h} [G_{j+\frac{1}{2}}^n - G_{j-\frac{1}{2}}^n]
\end{align*} \]

\( j \quad j+1 \quad j+\frac{1}{2} \quad G_{j+\frac{1}{2}}^n = G(U_j^n, U_{j+1}^n) \)

\( \text{numerical flux function} \)

\( G(U_j^n) = F(U_j^n) \) (consistency)

**Examples:**

**Lax-Friedrichs**

\( U_j^{n+1} = \frac{1}{2} (U_j^n + U_{j+1}^n) - \frac{\nu}{2} \left[ F_{j+1}^n - F_{j-1}^n \right] \)

\( (LF) \)

\( G(U_j^n, U_{j+1}^n) = \frac{1}{2} [F(U_j^n) + F(U_{j+1}^n)] - \frac{\nu}{2} (U_{j+1}^n - U_j^n) \)

**Lax-Wendroff**

\( (LW) \)

\( G = \frac{1}{2} \left( F_{j+1}^n + F_j^n \right) - \frac{\nu}{2} a_{j+\frac{1}{2}} \left( F_{j+1}^n - F_j^n \right) \)

**LWP smooths shocks too much**

**LW has too many oscillations**

\[ \begin{align*}
\text{approximation of next term to Taylor expansion} & \\
\text{next term} & = \frac{1}{2} (a_j + a_{j+1}) \\
a(n) & = f'(n)
\end{align*} \]
Breakthrough idea: Godunov's method

**REA algorithm: reconstruct - evolve - average**

1. **Reconstruct**: build a piecewise polynomial defined for all \( x \) from cell averages.

2. **Evolve**: solve the reconstructed problem exactly from \( t_n \) to \( t_{n+1} \) (for piecewise constant reconstruction). This involves solving a Riemann problem at each cell boundary.

- At \( t = t_n \):
  - \( u_L \) to the left of the boundary,
  - \( u_R \) to the right of the boundary.

- At \( t = t_{n+1} \):
  - \( u' \) at the new time step.

3. **Average**: replace \( u_j \) with the cell average.

\( j-\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ... \)
usually easier to evolve ad average via weak form of PDE:

\[
\frac{1}{\Delta x} \int_{x_j - \frac{1}{2}}^{x_j + \frac{1}{2}} u_j^\theta(x, t_n+1) \, dx = \frac{1}{\Delta x} \int_{x_j - \frac{1}{2}}^{x_j + \frac{1}{2}} u_j^\theta(x, t_n) \, dx + \frac{1}{\Delta x} \int_{x_j - \frac{1}{2}}^{x_j + \frac{1}{2}} \frac{F(u_j, t_n)}{\Delta t} \, dt
\]

\[
\frac{1}{\Delta x} \int_{x_j - \frac{1}{2}}^{x_j + \frac{1}{2}} \Delta x \cdot \psi(x, t_n+1) \, dx = \frac{1}{\Delta x} \int_{x_j - \frac{1}{2}}^{x_j + \frac{1}{2}} \Delta x \cdot \psi(x, t_n) \, dx + \frac{1}{\Delta x} \int_{x_j - \frac{1}{2}}^{x_j + \frac{1}{2}} \left[ \frac{F(u_j, t_n)}{\Delta t} \right] \, dt
\]

The Riemann problem at \( x = x_j - \frac{1}{2} \) and \( x = x_j + \frac{1}{2} \) are similarity solutions \( u = \phi \left( \frac{x - x_j - \frac{1}{2}}{t - t_n} \right) \), so they are constant along the rays \( x = x_j - \frac{1}{2} \) and \( x = x_j + \frac{1}{2} \).

Result:

\[
u_j^{n+1} = u_j^n + \Delta t \left[ \frac{\psi}{\Delta x} \left( \frac{F(u_j, t_n)}{\psi(u_j, t_n)} - F(u_j^{n+1}, t_n) \right) \right]
\]

(Leveque's notation for solution of Riemann problem along ray \( x = x_j - \frac{1}{2} \))

Automatically in conservative form:

\[
G(u_j, u_{j+1}) = F(u_j^{n+1}), \quad \text{left and right states in Riemann problem.}
\]
Usually \( U'(UL, UR) \) is just \( UL \) or \( UR \), (scheme reverts to upwind).

If \( F \) is convex \( (F'' > 0) \), the only exception is the transonic rarefaction, \( UL < U_S < UR \), where \( U_S \) satisfies \( a(U_S) = 0 \).

**Six cases:**

- Shock moving left \( U = UR \)
- Rarefaction moving left \( U = UR \)
- Transonic rarefaction \( U = U_S \)
- Stationary shock \( U = U_L \)
- Rarefaction moving right \( U = UR \)
- Shock moving right \( U = UL \)

In all cases,

\[
F^n_{j - \frac{1}{2}} = \begin{cases} 
\min_{u_{j-1} \leq u \leq u_j} F(u) & \text{if } u_{j-1} \leq U_j \\
\max_{u_{j-1} \geq u \leq u_j} F(u) & \text{if } u_{j-1} \geq U_j
\end{cases}
\]

This formula is also valid if \( F \) is concave \( (F''(x) < 0) \) or even in the nonconvex case where there could be several stagnation points.

**Note:** It's very important to treat the transonic rarefaction correctly, otherwise, Godunov can converge to the wrong weak solution.

Glimm's method (random choice method)

- Set up a piecewise constant Riemann problem \( U(x) = U_j \), \( x_{j-rac{1}{2}} \leq x \leq x_{j+\frac{1}{2}} \)
- Evolve exactly from \( t^n \) to \( t^{n+1} \).
- Instead of defining \( u_j^{n+1} \) to be the average of the exact solution over the cell, we now choose a random point in \( [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \) and use the value of the exact solution at that point for \( u_j^{n+1} \).
Last time:

\[ u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x} \left( C_{j+\frac{1}{2}} - C_{j-\frac{1}{2}} \right) \]

\[ C_{j+\frac{1}{2}} = \omega(u_j, u_{j+h}) \]

\[ g(u_j, u_{j+h}) = \min_{u_j \leq u \leq u_{j+h}} g(u) \quad \max_{u_j \leq u \leq u_{j+h}} g(u) \]

\[ U^*(u_L, u_R) = \text{exact solution of Riemann problem} \quad \text{at} \quad x = 0, t > 0 \]

(recall exact solution is of the form \( u(x, t) = \phi(x/t) \))

Glimm:

\[ u_j^{n+1} = U \left( u_{j-1}, u_j, u_{j+1}, \Delta x \right) \]

\[ U(u_L, u_0, u_R, \Delta x; x, t) = \text{exact solution with initial condition} \quad \Theta_j \in [0, 1] \]

Example:

\[ 0 < u_L < \xi \]

\[ 0 < u_L + u_R \]

\[ U(x, t) = \begin{cases} 
  u_L & 0 \leq x < u_L \Delta t \\
  \frac{u_L \Delta x - x}{(u_0 - u_L) \Delta t} u_L + \frac{x - u_L \Delta t}{(u_0 - u_L) \Delta t} u_0 & u_L \Delta t \leq x \leq u_0 \Delta t \\
  u_0 & u_0 \Delta t < x \leq \Delta x 
\end{cases} \]

Cases:

\[ u_L > u_0 \]

\[ \begin{cases} 
  s > 0 \quad s = \frac{u_L u_0}{2} & u_0 \leq u_R \\
  s \leq 0 & u_0 > u_R 
\end{cases} \]

\[ u_L < u_0 \]

\[ \begin{cases} 
  0 \leq u_L \leq u_0 \leq u_R & u_0 \leq u_R \\
  u_0 \leq u_R \leq u_0 \leq u_0 \Delta t & 0 \leq u_R \\
  u_0 \Delta t < x \leq \Delta x & u_0 < u_R \\
  u_0 \Delta t < x < \Delta x & u_0 < u_R 
\end{cases} \]
Convergence

Monotone scheme

\[ u_j^{n+1} = H \left( u_{j-q}^{n}, \ldots, u_{j+q}^{n} \right) \]

Requirement: \( H \) is an increasing function of each of its arguments (separately)

equivalently: \( \frac{\partial u_i^{n+1}}{\partial u_i^n} \geq 0 \quad \forall i, j \in \mathbb{Z} \)

Example: Lax-F \n\[ u_j^{n+1} = \frac{1}{2} \left( u_{j+1}^n + u_{j-1}^n \right) - \frac{\nu}{2} \left( F_{j+1} - F_{j-1} \right) \]

\[ \frac{\partial u_j^{n+1}}{\partial u_{j+1}^n} = \frac{1}{2} - \frac{\nu}{2} a(u_{j+1}^n) \geq 0 \quad \text{if} \quad \nu \leq \frac{1}{2a} \quad \text{(CFL)} \]

\[ \frac{\partial u_j^{n+1}}{\partial u_{j-1}^n} = \frac{1}{2} + \frac{\nu}{2} a(u_{j-1}^n) \geq 0 \quad \text{if} \quad \nu \leq \frac{1}{2a} \quad \text{(CFL)} \]

\[ \frac{\partial u_i^{n+1}}{\partial u_i^n} = 0 \quad \text{if} \quad i \notin \{j-1, j+1\} \]

Godunov's method is also monotone (can check all cases using

\[ u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x} \left( F(u_{j+\frac{1}{2}}) - F(u_{j-\frac{1}{2}}) \right) \]

Theorem: If a finite difference scheme in conservative

form is monotone along a refinement path (typically \( k = \nu \))

then the numerical solution converges to the vanishing

viscosity solution (i.e. satisfies the entropy condition)

as \( k \rightarrow 0 \).
Hemming proof in special case of Lax: \( v = \frac{1}{2} \)

\[
\begin{align*}
U^{n+1} & = \frac{1}{2} (U_{nn} + U_{n-1}) + \frac{v}{2} (F_{nn} - F_{n-1}) = 0
\end{align*}
\]

Q: What PDE are we really solving here?

Taylor expand \( u(x,t) \)

\[
\begin{align*}
u + ku_t + \frac{k^2}{2} u_{tt} & = \frac{1}{4} \left( (u + h u_t + \frac{h^2}{2} u_{tt})^{n+1} + \frac{v}{2} (2h) F(u)_{n+1} \right) \\
& + O(h^3)
\end{align*}
\]

\[
\begin{align*}
k (u_t + F(u) x) + \frac{k^2}{2} (u_{tt} - \frac{1}{v^2} u_{xx}) + O(h^3) & = 0
\end{align*}
\]

\[
\begin{align*}
U_t + F(u) x & = O(h) \\
U_{tt} & = -F(u) u_x + O(h) = \left( F'(u) u_t \right)_x + O(h)
\end{align*}
\]

so the numerical solution satisfies

\[
\begin{align*}
[U + F(u) x]_x & = \frac{k}{2} \left[ \left( \frac{1}{v^2} - a(u)^2 \right) U_{xx} \right] + O(h^2)
\end{align*}
\]

positive when CFL condition satisfied

RHS is like a viscous term \( \varepsilon U_{xx} \) but with a spatially varying diffusion constant.

expect to obtain the vanishing viscosity solution as \( k \to 0 \).

Unfortunately, monotone schemes are at most first order.
Consider the constant coefficient linear system

\[ \begin{align*}
U_t + A U_x &= 0 \\
U(x,0) &= U^0(x) \quad (F(u) = Au) \\
F'(w) &= DF(u)z = A 
\end{align*} \]

Diagonalize \( A = QAQ^{-1} \), \( \Lambda = \begin{pmatrix} \lambda_1 & & \\
& \ddots & \\
& & \lambda_m \end{pmatrix} \), \( Q = (q_1, \ldots, q_m) \)

\[ Q^{-1} = \begin{pmatrix} r_1^T \\
\vdots \\
r_m^T \end{pmatrix} \quad \text{and} \quad r_i^T q_j = \delta_{ij} \quad \text{(dual basis)} \]

Now solve decoupled scalar equations

\[ W = Q^{-1} u \]

\[ W_t + \Lambda W_x = 0 \]

\[ W(x,0) = W^0(x) = Q^{-1} U^0(x) \]

\[ W_i(x,t) = W_i^0(x-\lambda_i t) = r_i^T U^0(x-\lambda_i t) \]

\[ U(x,t) = Q W(x,t) = \sum_i q_i r_i^T U^0(x-\lambda_i t) \]

Same formula works if \( U^0(x) \) is discontinuous:

\[ U^0(x) = \begin{cases} U_L & x \leq 0 \\
U_R & x > 0 \end{cases} \]

\[ U(x,t) = \sum_{i=1}^m q_i r_i^T U^0(x-\lambda_i t) \]

\[ = \sum_{i:x<\lambda_i t} q_i r_i^T U_L + \sum_{i:x>\lambda_i t} q_i r_i^T U_R \]

Note that \( U(x,t) \) jumps by \( \alpha_i q_i \), where \( \alpha_i = r_i^T (U_R - U_L) \), when \( x \) crosses \( \lambda_i t \).

Also note that there are no rarefaction waves in the linear case.
In particular,

\[ u^\psi = u(0,t) = \sum_{i: \lambda_i > 0} q_i r_i^T u_L + \sum_{i: \lambda_i < 0} q_i r_i^T u_R \]

\[ F(u^\psi) = A u^\psi = \sum_{i: \lambda_i > 0} \lambda_i q_i r_i^T u_L + \sum_{i: \lambda_i < 0} \lambda_i q_i r_i^T u_R \]

\[ A_{q_i} = \lambda_i q_i \]

\[ = A^+ u_L + A^- u_R \]

Notation:

\[ \lambda^+ = \max(\lambda, 0) \quad \lambda^- = \min(\lambda, 0) \]

\[ \Lambda^+ = \begin{pmatrix} \lambda^+ & 0 \\ \lambda^+ & \lambda^+ \end{pmatrix} \quad \Lambda^- = \begin{pmatrix} \lambda^- & 0 \\ \lambda^- & \lambda^- \end{pmatrix} \]

\[ A^+ = \Theta \Lambda^+ \Theta^{-1} \quad \text{responsible for right-moving waves} \]

\[ A^- = \Theta \Lambda^- \Theta^{-1} \quad \text{left-moving waves} \]

\[ A = A^+ + A^- \quad |A| = A^+ - A^- = \Theta |\Lambda| \Theta^{-1} \]

Godunov reduces to upwind in this linear case:

\[ u_{i+1}^{n+1} = u_i^n + \frac{\Delta t}{\Delta x} \left[ (A^+ u_{j-1} + A^- u_j) - (A^+ u_j + A^- u_{j+1}) \right] \]

\[ = u_i^n - \frac{\Delta t}{\Delta x} \left[ A^+ \Delta u_{j-\frac{1}{2}} + A^- \Delta u_{j+\frac{1}{2}} \right] \]

Upwind:

\[ a > 0: \quad u_{i+1}^{n+1} = u_i^n - a \nu (u_j^n - u_{j-1}^n) \]

\[ a < 0: \quad u_{i+1}^{n+1} = u_i^n - a \nu (u_{j+1}^n - u_j^n) \]

Either way:

\[ u_{i+1}^{n+1} = u_i^n - \nu \left( a^+ \Delta u_{j-\frac{1}{2}} + a^- \Delta u_{j+\frac{1}{2}} \right) \]
High resolution methods
try to improve order of accuracy in smooth regions
while keeping shocks sharp and avoid oscillation

key idea: rework Lax-Wendroff flux

\[
G_i^{n-\frac{1}{2}} = \frac{1}{2} A (u_i^{n-1} + u_i^n) - \frac{1}{2} \frac{\Delta t}{\Delta x} A^+ (u_i^n - u_i^{n-1})
\]

unstable centered flux

\[
G_i^{n-\frac{1}{2}} = (A^+ u_i^{n-1} + A^- u_i^n) + \frac{1}{2} |A| \left( I - \frac{\Delta t}{\Delta x} |A| \right) (u_i^n - u_i^{n-1})
\]

stable upwind flux

2nd order correction

here we used |A| = A^+ - A^- (just expand it all out)
and collect terms

now we choose a flux limiter function \( \phi(\theta) \)

with the idea that \( \phi(\theta) = 1 \Rightarrow 2\text{nd order} \)
\( \phi(\theta) = 0 \Rightarrow 1\text{st order} \)

(scalar case)

scheme: multiply correction term by \( \phi\left(\theta_i^{n-\frac{1}{2}}\right) \)

\[
\theta_i^{n-\frac{1}{2}} = \frac{\Delta u_i^{n-1}}{\Delta x} \quad I = \left\{ i-1, a_{i-\frac{1}{2}} > 0 \right\} \quad \text{upwind direction}
\]

in smooth regions, \( \theta \approx 1 \); near shocks, \( \theta \) can have any value.

upwind method: \( \phi(\theta) = 0 \)

Lax-Wendroff method: \( \phi(\theta) = 1 \)

\[
\minmod(u) = \begin{cases} 0 & \text{if } a \\ \min_i u_i & \text{if } a \end{cases}
\]

\[
\superl(u) = \begin{cases} \max(0, \min(-1, \theta), \min(1, \theta)) & \text{if } a < 0 \\ \theta & \text{if } a \end{cases}
\]

\[
MC = \begin{cases} \max(0, \min(1 + \theta i, 2, 20)) & \text{if } a < 0 \\ \theta & \text{if } a \end{cases}
\]

the latter 3 schemes are TVD = total variation diminishing (no oscillations)
(continuation from 228A notes)

Last time: double-layer \( u(x) = \int_T -\frac{\partial N}{\partial n_x}(x, \xi) \phi(\xi) \, d\xi \)

single-layer \( u(x) = \int_T N(x, \xi) \phi(\xi) \, d\xi \)

4 types of B.C.'s:

- **Interior Dirichlet**: \( u = 0 \)
- **Exterior Dirichlet**: \( n \cdot \nabla u = 0 \)
- **Interior Neumann**: \( \frac{\partial u}{\partial n} = g \)
- **Exterior Neumann**: \( u = g \)

The integral equations of potential theory are:

- **Interior Dirichlet**: \( \left( \frac{1}{2} \mathbb{I} + iK \right) \phi = g \)
- **Exterior Dirichlet**: \( \left( -\frac{1}{2} \mathbb{I} + iK \right) \phi = g \)
- **Interior Neumann**: \( \left( \frac{1}{2} \mathbb{I} + iK^* \right) \phi = -g \)
- **Exterior Neumann**: \( \frac{\partial u}{\partial n} = g \)

\( g = \frac{\partial u}{\partial n} = \nabla u \cdot n \), \( n \) outward normal from \( \partial \Omega \) in both cases.

\[ K \phi(x) = \int_T K(x, \xi) \phi(\xi) \, d\xi \quad \text{order reversed (it's like a transpose)} \]

\[ K^* \phi(x) = \int_T K(\xi, x) \phi(\xi) \, d\xi \]
\[ K(x, \bar{x}) = \left\{ \begin{array}{ll}
\frac{1}{2\pi} \frac{d\theta}{d\tau} (x, \bar{\tau}) & \quad \bar{x}, \bar{\tau} \in \Gamma, \bar{x} \neq \bar{\tau} \\
\frac{1}{4\pi} \kappa(x) & \quad \bar{x} = \bar{\tau} \in \Gamma
\end{array} \right. \]

\[ \frac{d\theta}{ds} = \frac{(x-x') \eta' - (\eta-\eta') x'}{(x-x')^2 + (\eta-\eta')^2} \cdot \frac{1}{\sqrt{(x')^2 + (\eta')^2}} \]

\[ \kappa = \frac{x'' - x' \eta''}{[(x')^2 + (\eta')^2]^{3/2}} = \text{curvature} \]

\[ K > 0 \quad K < 0 \]

IK and IK* are adjoints in \( L^2(\Gamma) \)

\[ \langle IK \phi, \psi \rangle = \langle \phi, IK^* \psi \rangle \quad \forall \phi, \psi \in L^2(\Gamma) \]

\[ \langle \phi, \psi \rangle = \int_{\Gamma} \phi(z) \overline{\psi(z')} ds \]

They are also both compact operators (almost finite rank)

Fredholm alternative: Suppose \( A : L^2(\Gamma) \to L^2(\Gamma) \) has the form \( A = \alpha \mathbb{I} + K \), \( \alpha \in \mathbb{C}, \alpha \neq 0 \), IK compact

Thus either:

1. \( A \phi = \gamma \) and \( A^* \psi = \gamma \) have unique solutions for all \( \phi, \psi \in L^2(\Gamma) \)
2. \( A \phi = \gamma \) is solvable iff \( \langle \gamma, \psi \rangle = 0 \) \( \forall \psi \in \mathcal{N}(A^*) \) and \( \mathcal{N}(A) = \mathcal{N}(A^*) \) and they have the same (finite) dimension
The same thing happens in finite dimensions.

Example:
\[
\begin{bmatrix}
3 & 2 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
A^* = \begin{bmatrix}
3 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\end{bmatrix}
N(A^*) = \text{span}(w), w = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}
N(A) = \text{span}(e_3), e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

Is there a solution of \(Ax = b, b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\)?
- Yes, \(b^Tw = 0\)
- \(b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\), \(w_3 = 1 \neq 0\)

The Fredholm alternative reduces the question of solvability (for which \(y\) is there a solution?) to one of uniqueness (if \(x = 0\), how many solutions are there?). But, this second question pertains to the adjoint problem:

\[
\begin{align*}
\text{interior Dirichlet} & \iff \text{exterior Neumann} \\
\text{exterior Neumann} & \iff \text{interior Dirichlet}
\end{align*}
\]

Dimensions of Kernels: (Following Folland's PDE book)

Suppose \(\Omega\) has \(m\) connected components \(\Omega_1, \ldots, \Omega_m\)
\[
\Omega, \Omega_1, \ldots, \Omega_m
\]

Example:

\[
\begin{align*}
m &= 2 \\
m' &= 1
\end{align*}
\]
Let \( \phi_i(x) = \begin{cases} 1 & x \in \mathbb{S}^i; 
0 & \text{o.w.} \end{cases} \) \((i = 1, \ldots, m)\)

\( \psi_i(x) = \begin{cases} 1 & x \in \mathbb{S}^{i-1}; 
0 & \text{o.w.} \end{cases} \) \((i = 1, \ldots, m'; i = 0 \text{ is excluded})\) (on purpose)

Claim: \( V = \ker (-\frac{1}{2} \Pi + 1k) = \text{span}\{ \phi_i \} \)

\( W = \ker (\frac{1}{2} \Pi + 1k) = \text{span}\{ \psi_i \} \)

Claim: \( (m, 2b) \) reason: winding numbers cancel as you cross boundaries.

Diagram:

1. \( \phi_i \)
2. \( k \phi_i \)
3. \( \psi_i \)
4. \( K \psi_i \)
5. \( \psi_2 \)
so \( \text{dim } V \geq m \), \( \text{dim } W' \geq m' \). Want to show:

Let \( V_1 = \ker ( -\frac{1}{2} \mathbb{I} + 1K^*) \), \( W_1 = \ker ( \frac{1}{2} \mathbb{I} + 1K^*) \)

by F.A., \( \text{dim } V_1 = \text{dim } V \), \( \text{dim } W_1 = \text{dim } W \).

Claim: \( \text{dim } V_1 \leq m \), \( \text{dim } W_1 \leq m' \).

If \( \phi \in V_1 \), then \( u(x) = \int N(x, \xi) \phi(\xi) d\xi \)

Solve the interior Neumann problem \( \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega. \end{cases} \)

so \( u \) is constant on each connected component of \( \Omega' \).

Thus, we have a linear mapping \( \beta \) from \( \phi \in V_1 \) to \( \beta(\phi) = \left( u_{j_1}, \ldots, u_{j_m} \right) \in \mathbb{R}^m \).

This mapping is injective (in 3d) since single layer potentials are continuous across \( \partial \Omega \), and if \( u = 0 \) on each \( \Omega_j \), then \( u \) solves the exterior Dirichlet problem with 0 boundary conditions.

In 3d, this implies \( u = 0 \) on \( \Omega' \) as well, so \( \phi = \frac{\partial u^-}{\partial n} - \frac{\partial u^+}{\partial n} = 0 \).

In 2d, you instead show that \( \ker(\beta) \) is one-dimensional and \( \text{ran}(\beta) \) has codimension 1.
complex variable methods in 2d potential theory

\[ \Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \Gamma \]

\[ u(\xi) = \int_\Gamma -\frac{dN}{d\xi}(\xi, \xi') \, \mu(\xi') \, ds' \]

Using \( \mu \) instead of \( \phi \) today

\[
\begin{aligned}
 &\xi = (x, y) \rightarrow \xi = x + iy \\
 &\xi' = (\xi, \eta) \rightarrow \xi = \xi + \gamma \\
 \end{aligned}
\]

- curve parametrized by \( \gamma(\alpha), 0 \leq \alpha \leq \alpha \)

\[
- \frac{dN}{d\xi} \, ds = \frac{1}{2\pi} \, d\theta \, d\alpha = \frac{1}{2\pi} \, \frac{(\eta(\alpha) - x, \eta(\alpha) - y) \cdot (\eta'(\alpha) - \xi'(\alpha))}{(\xi - \eta)^2 + (\eta - y)^2} \, d\alpha
\]

\[
\begin{aligned}
 &\frac{(a - b)(x - y)}{a^2 + b^2} = \frac{ay - bx}{a^2 + b^2} = \text{Im} \left( \frac{(a - b)(x + iy)}{a^2 + b^2} \right) = \text{Im} \left( \frac{x + iy}{a + ib} \right) \\
 \end{aligned}
\]

\[
- \frac{dN}{d\eta} \, ds = \text{Im} \left\{ \frac{1}{2\pi} \, \frac{\xi'(\alpha)}{\xi(\alpha) - \xi} \, d\alpha \right\} = \text{Re} \left\{ \frac{1}{2\pi i} \, \frac{\xi'(\alpha)}{\xi(\alpha) - \xi} \, d\alpha \right\}
\]

double layer potential becomes

\[ u(z) = \text{Re} \left\{ \frac{1}{2\pi i} \int_0^\alpha \frac{\xi'(\alpha)}{\xi(\alpha) - z} \, \mu(\alpha) \, d\alpha \right\} \]

\[ u(z) = u(z) + i\, v(z) = \frac{1}{2\pi i} \int_0^\alpha \frac{\xi'(\alpha)}{\xi(\alpha) - z} \, \mu(\alpha) \, d\alpha \quad \text{Cauchy Integral} \]

When \( z = \xi(\alpha) \) is on the boundary, this is a singular integral, which has to be interpreted in the principal value sense:

\[ u(\xi(\alpha)) = \frac{1}{2\pi i} \int_0^\alpha \frac{\xi'(\beta)}{\xi(\alpha) - \xi(\beta)} \, \mu(\beta) \, d\beta \]
\[ \lim_{\varepsilon \to 0} \left( \int_{0}^{a-\varepsilon} + \int_{a+\varepsilon}^{a} \right) \]

Remove symmetric interval \([a-\varepsilon, a+\varepsilon]\) from integration domain.

Plancherel formula for Cauchy integrals:

\[
\mathcal{U}(\delta(\alpha) \pm) = \pm \frac{1}{2} \mu(\alpha) + \frac{1}{2\pi i} \int_{0}^{a} \frac{\delta'(\beta)}{\delta(\beta) - \delta(\alpha)} \mu(\beta) d\beta
\]

Special case: \(\delta(\alpha) = \alpha\), \(\alpha \in \mathbb{R}\) (real line instead of periodic curve)

\[
\mathcal{U}(\varepsilon) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\mu(\alpha)}{\alpha - \varepsilon} d\alpha, \quad \text{Re}\{\varepsilon\} > 0
\]

\[
\mathcal{U}(x \pm) = \pm \frac{1}{2} \mu(x) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\mu(\beta)}{\beta - x} d\beta
\]

Hilbert transform: given \(f(x) = \text{Re}\{\mathcal{U}(x^+)\}\) return \(Hf(x) = \text{Im}\{\mathcal{U}(x^+)\}\)

Formula: \(Hf(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\beta)}{x - \beta} d\beta\) \(\left(\frac{1}{i} \frac{1}{\beta - x} = \frac{i}{x - \beta}\right)\)

When \(f(x)\) is periodic, \(\int_{-\infty}^{\infty} \cdots \) means \(\lim_{N \to \infty} \left( \int_{-N}^{-\varepsilon} + \int_{N}^{N} \cdots \right)\)

Write \(f(x) = \sum_{k} \hat{f}_{k} e^{ikx}\), where \(\hat{f}_{-k} = \overline{\hat{f}_{k}}\)

The extension \(\mathcal{U}(\varepsilon) = \sum_{k} \hat{f}_{k} e^{ik\varepsilon}\) is not bounded as \(y \to \infty\)

But \(\mathcal{U}(\varepsilon) = f_{0} + 2 \sum_{k=1}^{\infty} \hat{f}_{k} e^{ik\varepsilon}\) is fine (since \(e^{ik(x+iy)} = \overline{e^{ikx}} e^{-k y}\) for \(y \to \infty\))
\[
\text{Re}\{U(x)\} = f_0 + \sum_{k=1}^{\infty} \left[ \hat{f}_k e^{ik\tau} + \bar{\hat{f}}_k e^{-ik\tau} \right] = f(x)
\]

\[
\text{Im}\{U(x)\} = \sum_{k=1}^{\infty} \left[ -i \hat{f}_k e^{ik\tau} + i \bar{\hat{f}}_k e^{-ik\tau} \right] = Hf(x)
\]

So \( Hf_k = \begin{cases} -i \hat{f}_k & k > 0 \\ 0 & k = 0 \\ i \bar{\hat{f}}_k & k < 0 \end{cases} \)

Here we used \( \text{Re } w = \frac{w + \bar{w}}{2} \), \( \text{Im } w = \frac{w - \bar{w}}{2i} \).

Alternatively, when \( f(x) \) is periodic, we can write

\[
Hf(x) = \frac{1}{\pi} \int_{0}^{2\pi} p.v. \sum_{k=-\infty}^{\infty} \frac{f(\beta)}{x-(\beta+2\pi k)} \, d\beta
\]

Principal value sum \( p.v. \sum_{k=-\infty}^{\infty} a_k = a_0 + \sum_{k=1}^{\infty} [a_k + a_{-k}] \).

Euler: \( \sin z = \prod_{k=1}^{\infty} \left( 1 - \frac{z^2}{n^2\pi^2} \right) = 2 \, p.v. \prod_{k=0}^{\infty} \left( 1 - \frac{z^2}{n\pi} \right) \)

\[
\log \sin z = \log z + p.v. \sum_{k=0}^{\infty} \log \left( 1 - \frac{z^2}{n\pi} \right)
\]

\[
\frac{d}{dz} \left( \cot z \right) = \frac{1}{z} + p.v. \sum_{k=0}^{\infty} \frac{1}{z-\pi k} = p.v. \sum_{k}^{\infty} \frac{1}{z-\pi k}
\]

\[
\frac{1}{2} \cot \frac{z}{2} = p.v. \sum_{k}^{\infty} \frac{1}{z-2\pi k}
\]

So \( Hf(x) = \frac{1}{\pi} \int_{0}^{2\pi} \frac{f(\beta)}{2} \cot \left( \frac{x-\beta}{2} \right) \, d\beta \)
We can use this form of the Hilbert transform to regularize a general Cauchy integral:

\[ U(\xi(x)) = \pm \frac{1}{2} \mu(x) + \frac{1}{2\pi i} \int_0^a \frac{\xi'(\beta)}{\xi(\beta) - \xi(x)} \mu(\beta) \, d\beta \]

Suppose \( a = 2\pi \) for simplicity. Then

\[ \frac{1}{2\pi i} \int_0^{2\pi} \frac{\xi'(\beta)}{\xi(\beta) - \xi(x)} \mu(\beta) \, d\beta = \frac{1}{2\pi i} \int_0^{2\pi} \left[ \frac{\xi'(\beta)}{\xi(\beta) - \xi(x)} + \frac{1}{2} \cot \left( \frac{\beta - x}{2} \right) \right] \mu(\beta) \, d\beta \]

Up.

Just a Riemann integral now

\[ \text{since} \quad \lim_{\beta \to x} \left[ \ldots \right] = \frac{\xi''(x)}{2} \xi'(x) \]

(the singular leading terms cancel)

\[ \text{still a principal value integral, but a particularly nice one} \quad \left( \text{it equals} \quad \frac{i}{2} \text{H} \mu(x) \right) \]

We can evaluate it using the FFT!

When we take the real part of \( U(\xi(x)) \), the regularizing terms disappear (as they are imaginary) and we have

\[ \frac{1}{2} \mu(x) + \frac{1}{2\pi} \int_0^{2\pi} K(\alpha, \beta) \mu(\beta) \, d\beta = g(x) \]

\[ K(\alpha, \beta) = \text{Re} \left\{ \frac{\xi'(\beta)}{\xi(\beta) - \xi(x)} \right\} = \left\{ \begin{align*}
& \text{Im} \left\{ \frac{\xi'(\beta)}{\xi(\beta) - \xi(x)} \right\} \quad \alpha \neq \beta \\
& \text{Im} \left\{ \frac{\xi''(\beta)}{2\xi'(\beta)} \right\} \quad \alpha = \beta
\end{align*} \right\} \]

matrix form: \(( \frac{1}{2} I + K ) \mu = g \), \( K_{kj} = \frac{1}{N} K(\alpha_k, \beta_j) \), \( \alpha_k = \frac{2\pi k}{N} \), \( \beta_j = \frac{2\pi j}{N} \)
Next we want to compute the normal derivative of $U$ on the boundary. For any function $u(z)$, we have

$$
\hat{n} \frac{du}{dn} = (u_x, u_y) \cdot \frac{(\eta' - \xi')}{|\xi'|} = \text{Im} \left\{ \frac{(u_x - iu_y)(\xi' + i\eta')}{|\xi'|} \right\}
$$

If $U(z) = u(z) + iv(z)$ is analytic, then

$$
u_x = u_y, \quad u_y = -v_x \quad \text{(Cauchy–Riemann equations)}
$$

so

$$
\frac{du}{dn} = \text{Im} \left\{ \frac{(u_x + iv_x)(\xi')}{|\xi'|} \right\} = \text{Im} \left\{ \frac{Uz' \xi'}{|\xi'|^2} \right\}
$$

This formula could be used for the exact solution in the homework:

$$
U(z) = 2 \log |z - (i - z)| \Rightarrow \frac{du}{dn}(5(\alpha)) = \text{Im} \left\{ \frac{2}{\alpha - (i - 2)} - \frac{5'(\alpha)}{|5'(\alpha)|} \right\}
$$

It can also be used for the numerical solution:

$$
U(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{5'(\alpha)}{5(\alpha) - z} M(\alpha) d\alpha
$$

$$
\frac{du}{dn}(z, \alpha) = \nabla u(z) \cdot \hat{n}_{5(\alpha)} = \text{Im} \left\{ \left( \frac{1}{2\pi i} \int_0^{2\pi} \frac{5'(\beta)}{(5(\beta) - z)^2} M(\beta) d\beta \right) \frac{5'(\alpha)}{|5'(\alpha)|} \right\}
$$

Note that

$$
\frac{5'(\beta)}{(5(\beta) - z)^2} = -\frac{1}{\beta - z} \Rightarrow \frac{1}{5(\beta) - z}
$$

so we can integrate by parts to get

$$
\frac{du}{dn}(z, \alpha) = \text{Im} \left\{ \left( \frac{1}{2\pi i} \int_0^{2\pi} \frac{M(\beta)}{5(\beta) - z} d\beta \right) \frac{5'(\alpha)}{|5'(\alpha)|} \right\}
$$
We now define \( \frac{\partial u}{\partial n}(S(a)^\pm) = \lim_{z \to S(a)^\pm} \frac{\partial u}{\partial n}(z, \alpha) \) which gives
\[
\left| S'(a) \right| \frac{\partial u}{\partial n}(S(a)^\pm) = \text{Im} \left\{ \lim_{z \to S(a)^\pm} \frac{1}{2 \pi i} \int_0^{2\pi} \frac{S'(a)}{S(b)-z} \mu'(\beta) \, d\beta \right\}
\]

This limit can be evaluated by writing
\[
\frac{S'(a)}{S(b)-z} = \frac{S'(a)-S'(b)}{S(b)-S(a)} + \frac{S'(b)}{S(b)-S(a)}
\]

result is
\[
\left| S'(a) \right| \frac{\partial u}{\partial n}(S(a)^\pm) = \text{Im} \left\{ \pm \frac{1}{2} \mu'(\alpha) + \frac{1}{2 \pi i} \int_0^{2\pi} \frac{S'(a)}{S(b)-S(a)} \mu'(\beta) \, d\beta \right\}
\]

so \( \frac{\partial u}{\partial n}(z, \alpha) \) is actually continuous (in \( z \)) across \( \Gamma \) since the jump term is real.

We use our regularization trick to evaluate the principal value integral:
\[
\left| S'(a) \right| \frac{\partial u}{\partial n} = \text{Im} \left\{ \frac{1}{2 \pi i} \int_0^{2\pi} \left[ \frac{S'(a)}{S(b)-S(a)} - \frac{1}{2} \cot \left( \frac{\beta-\alpha}{2} \right) \right] \mu'(\beta) \, d\beta \right\}
\]

or
\[
\left| S'(a) \right| \frac{\partial u}{\partial n}(S(a)) = \frac{1}{2} \text{Re} \left[ \mu'(\alpha) \right] + \frac{1}{2 \pi i} \int_0^{2\pi} \frac{S'(a)}{2 S(a)} \mu'(\beta) \, d\beta
\]

\[
\Gamma(\alpha, \beta) = \begin{cases} \text{Re} \left\{ \frac{S'(a)}{S(a)-S(b)} - \frac{1}{2} \cot \left( \frac{\beta-\alpha}{2} \right) \right\} & \alpha \neq \beta \\ \text{Re} \left\{ \frac{S''(a)}{2 S(a)} \right\} & \alpha = \beta \\ \end{cases}
\]

in matrix form:
\[
\frac{\partial u}{\partial n} = \left( \frac{1}{2} \text{HD} \mu + \Gamma \text{D} \mu \right)
\]

\( \Gamma_{ij} = \frac{1}{N} \left| S'(a) \right| \Gamma(\alpha_i, \beta_j) \)
The FFT can be used to apply $H$ and $D$ without actually forming these matrices:

\[
Df = \mathcal{F}^{-1} \text{diag}(\varepsilon^{-1} \Xi_k) \mathcal{F} f, \quad Hf = \mathcal{F}^{-1} \text{diag}(-i \text{sgn}(k)) \mathcal{F} f
\]

\[
\Xi_k = \frac{2\pi}{N} \left\{ \begin{array}{ll}
1, & 1 \leq k \leq \frac{N}{2} \\
0, & \frac{N}{2} + 2 \leq k \leq N \\
\varepsilon, & k = \frac{N}{2} + 1
\end{array} \right.
\]

\[
\text{sgn}(k) = \begin{cases}
1, & 2 \leq k \leq \frac{N}{2} \\
-1, & \frac{N}{2} + 2 \leq k \leq N \\
0, & k = 1 \text{ or } k = \frac{N}{2} + 1
\end{cases}
\]

---

Multiply-connected domains

Suppose $\Sigma$ has one connected component and $\Sigma' = \mathbb{R}^2 \setminus \Sigma$ has $m'$ connected components.

\[
\includegraphics[width=0.2\textwidth]{multiply_connected_domain}
\]

The operator $\frac{1}{2} \Pi + IK$ has a kernel of dimension $m'$.

Specifically, the functions

\[
I_k(x) = \begin{cases}
1, & x \in T_k \\
0, & x \in T_j, \ j \neq k
\end{cases}
\]

$m' = 3$ example

are a basis for $\ker(\frac{1}{2} \Pi + IK)$

By the Fredholm alternative, $\frac{1}{2} \Pi + IK^*$ also has a kernel, $V^*$, of dimension $m'$, and

\[
B\mu = \left(\frac{1}{2} \Pi + IK\right) \mu = \mathfrak{g}
\]

is solvable if $\langle \mathfrak{g}, \phi \rangle = 0 \ \forall \phi \in \ker(\frac{1}{2} \Pi + IK^*)$
The solution is to modify the layer potential so that the resulting integral equation takes the form

\[ \tilde{B} \mu = g, \quad \tilde{B} = B + \sum_{k=1}^{m'} \langle \cdot, I_k \rangle y_k, \]

where span\{\{y_1, \ldots, y_m\}\} is any complement of \((V^*)^\perp\) in \(L^2(\Gamma)\). range of \(B\).

I claim the functions \(y_j(x) = \log |x - \tilde{a}_j|\), \(1 \leq j \leq m'\), \(x \in \Gamma\) work, where \(\tilde{a}_j\) is an arbitrary point inside \(T_j\). Suppose to the contrary that a nonzero linear combination \(g = \sum_j \alpha_j y_j\) belongs to \((V^*)^\perp\). Then \(\text{ran}(B)\), so the solution of the Dirichlet problem \(\{\Delta u = 0 \text{ in } \Omega; \quad u = g \text{ on } \Gamma\}\) can be represented using a double-layer potential. But the solution of this problem is \(u(x) = \sum_j \alpha_j \log |x - \tilde{a}_j|\) for \(x \in \Omega\). Since the normal derivative \(\frac{du}{dn}\) of a double layer potential is continuous across each \(T_i\), the function \(u(x)\) satisfies the Neumann problem

\[ \Delta u = 0 \text{ in } \Omega; \]
\[ \frac{du}{dn} = \tilde{g}_i \text{ on } T_i = \partial \Omega_i; \]

where \(\tilde{g}_i(x) = \sum_j \alpha_j \frac{2}{n} \log |x - \tilde{a}_j|\) for \(x \in T_i\).

In other words, the data for the Neumann problem in the \(i\)th hole comes from the normal derivative of the solution just outside the hole.
But the 1-th Neumann problem has a solution iff \( \sum s_i = 0 \).
This follows from the divergence theorem:

\[
\int_{T_i} \mathbf{v} \cdot \mathbf{n} \, ds = \int_{\Omega_i \setminus B_i} \nabla \cdot \mathbf{v} \, dx = 0
\]

Since the function \( w(x) = \sum_{j} \alpha_j \log |x - a_j| \) satisfies \( \Delta w = 0 \) in \( \Omega \setminus \bigcup B_i \), we can apply the same argument to conclude

\[
\int_{T_i} \frac{\partial w}{\partial n} \, ds = \int_{\partial B_i} \frac{\partial w}{\partial n} \, ds = \int_0^{2\pi} \alpha_j \, d\theta = -2\pi \alpha_j
\]

only the \( j \)-th term contributes centered at \( a_j \), so all the \( \alpha_j \) must be zero which is a contradiction.

Going back to complex variables and parametrizing each curve \( T_i \) separately by its own function \( \gamma_i(\alpha) \), the new representation is

\[
U(z) = \sum_{j=0}^{m'} \frac{1}{2\pi i} \int_0^{2\pi} \frac{\gamma_i'(\alpha)}{\gamma_i(\alpha) - z} \mu_j(\alpha) \, d\alpha
\]

\[
+ \sum_{j=1}^{m'} \log (z - a_j) \cdot \frac{1}{2\pi} \int_0^{2\pi} \mu_j(\alpha) \, d\alpha
\]

\[
U(z) = u(z) + iv(z)
\]

\[
u(z) = \text{Re} \{ U(z) \}
\]

\[
P_0 \mu_j
\]

is an operator that computes the average value of a function.
The integral equations we need to solve are:

\[
\frac{1}{2} \mu_k(\alpha) + \sum_{j=0}^{m'} \frac{1}{2\pi} \int_0^{2\pi} \left[ K_{k,j}(\alpha,\beta) + C_{k,j}(\alpha) \right] \sqrt{\mu_j(\beta)} d\beta = g_k(\alpha)
\]

\[
K_{k,j}(\alpha,\beta) = \begin{cases} 
\text{Im} \left\{ \frac{\delta_j'(\beta)}{\delta_j(\beta) - \delta_k(\alpha)} \right\} & \text{if } k \neq j \text{ or } \alpha \neq \beta \\
\text{Im} \left\{ \frac{\delta_k'(\alpha)}{2\delta_k(\alpha)} \right\} & \text{if } k = j \text{ and } \alpha = \beta
\end{cases}
\]

\[
C_{k,j}(\alpha) = \begin{cases} 
\log |\delta_k(\alpha) - a_j| & j = 1, \ldots, m' \\
0 & j = 0
\end{cases}
\]

\[
1(\beta) = 1
\]

The matrix version when \( m' = 2 \) is:

\[
\left[ \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} K_{00} & K_{01} & K_{02} \\ K_{10} & K_{11} & K_{12} \\ K_{20} & K_{21} & K_{22} \end{pmatrix} + \begin{pmatrix} C_{00} & C_{01} & C_{02} \\ C_{10} & C_{11} & C_{12} \\ C_{20} & C_{21} & C_{22} \end{pmatrix} \right] \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}
\]

where

\[
(K_{k,j})_{lm} = \frac{1}{\eta_j} K_{k,j}(\alpha_l, \beta_m)
\]

\[
(C_{k,j})_{lm} = \frac{1}{\eta_j} C_{k,j}(\alpha_l) 1(\beta_m)
\]

curve \( T_j \) has \( \eta_j \) collocation points.

(The rows of the matrix represent trapezoidal rule approximations of the corresponding integrals.)
To compute the normal derivative of the solution, we use the formula

\[
|S_h(\alpha)| \frac{\partial u}{\partial n}(S_h(\alpha)) = \frac{1}{2} H[M'(\alpha)](\alpha) + \sum_{j=0}^{m'} \frac{1}{2\pi} \int_0^{2\pi} \sigma_{k_j}(\alpha, \beta) \mu'(\beta) d\beta
\]

\[+ \sum_{j=1}^{m'} \left\{ \frac{S_h'(\alpha)}{S_h(\alpha) - \alpha_j} \right\} p_0 \mu_j \]

where

\[
\sigma_{k_j}(\alpha, \beta) = \begin{cases} 
\text{Re} \left\{ \frac{S_h'(\alpha)}{S_h(\alpha) - S_j(\beta)} - \frac{1}{2} \cot(\frac{\alpha - \beta}{2}) \right\} & \text{if } k = j, \alpha \neq \beta \\
\text{Re} \left\{ \frac{S_h''(\alpha)}{2S_h'(\alpha)} \right\} & \text{if } k = j, \alpha = \beta \\
\text{Re} \left\{ \frac{S_h'(\alpha)}{S_h(\alpha) - S_j(\beta)} \right\} & \text{if } k \neq j
\end{cases}
\]
water waves

setup: free surface $\eta(x, t)$ evolving in time

Euler equations (incompressible, irrotational, inviscid)

$$\begin{align*}
p \left( \dot{\mathbf{u}} + \mathbf{u} \cdot \nabla \mathbf{u} \right) &= -\nabla p - \rho g \hat{y} \\
p \cdot \mathbf{u} &= 0
\end{align*}$$

$\mathbf{u}$ = velocity
$p$ = pressure
$\rho$ = density
$g$ = gravitational acceleration

assumption: vorticity is zero

$\Rightarrow$ there is a potential function $\phi$ such that $\mathbf{u} = \nabla \phi$

(assuming the domain is simply-connected)

since $\nabla \left( \frac{1}{2} \| \mathbf{u} \|^2 \right) = \nabla \left( \frac{1}{2} u^2 + \frac{1}{2} v^2 \right) = \begin{pmatrix} uu_x + vv_x \\ uu_y + vv_y \end{pmatrix}$

and $\mathbf{u} \cdot \nabla \mathbf{u} = (u \partial_x + v \partial_y) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} uu_x + vv_x \\ uu_y + vv_y \end{pmatrix}$

they are equal

we see that if the flow is irrotational ($\nabla \times \mathbf{u} = (V_x - U_y) \hat{z} = 0$)

$\Rightarrow$ becomes

$$\nabla \left( \phi_t + \frac{1}{2} \| \nabla \phi \|^2 + \frac{p}{\rho} + gy \right) = 0$$

$\Delta \phi = 0$

so $\phi_t + \frac{1}{2} \| \nabla \phi \|^2 + \frac{p}{\rho} + gy = C(t)$

constant in space, allowed to vary in time

unsteady Bernoulli equation
Consider a particle \((x(t), y(t))\) on the free surface.

\[
y(t) = \eta(x(t), t)
\]

\[
y = \eta_x \dot{x} + \eta_t
\]

\[
\nu = \eta_x u + \eta_t
\]

We also define \(\Phi(x, t) = \phi(x, \eta(x, t), t)\) and check that

\[
\frac{\partial \Phi}{\partial t} = \phi_y \eta_t + \phi_t
\]

Using the Bernoulli equation to evaluate \(\phi_t\), we get

\[
\frac{\partial \Phi}{\partial t} = \phi_y \eta_t - \frac{1}{2} \phi_x^2 - \frac{1}{2} \phi_y^2 - \frac{P}{\rho} - gy + c(t)
\]

We usually choose \(c(t)\) so that \(\int_{a}^{b} \frac{\partial \Phi(x, t)}{\partial x} \, dx = 0\).

Final set of equations:

**Initial conditions:** \(\eta(x, 0) = \eta_0(x), \quad \Phi(x, 0) = \Phi_0(x)\) \((t = 0)\)

have to solve

Laplace equation to obtain \(\phi_x, \phi_y\)
on boundary

actual evolution equations

\[
\frac{\partial \Phi}{\partial t} = \Delta \Phi - \frac{1}{2} \nabla \phi^2 - gy
\]

\((y = \eta)\)

\[
\eta_t = \phi_y - \eta_x \phi_x
\]

\((y = \eta)\)

\[
\eta_t = \phi_y - \eta_x \phi_x
\]

\((y = \eta)\)

\(P = 0\) at free surface

project out the mean

\(\bar{f}(x) = \int_{-\infty}^{\infty} f(x) \, dx\)

define

\[
\nabla \Phi = \phi_y - \eta_x \phi_x = \sqrt{1 + \eta_x^2} \frac{\partial \phi}{\partial y}
\]

\(\Delta \Phi = \phi_x + \eta_x \phi_x = \sqrt{1 + \eta_x^2} \frac{\partial \phi}{\partial x} - \frac{1}{2} \nabla \phi^2 - gy\)

\(-\text{Dirichlet-Neumann operator}\)

so

\[
(\nabla \Phi) = (1 - \eta_x) (\phi_y) \quad \text{and} \quad (\phi_x) = \frac{1}{1 + \eta_x^2} \left(1 - \eta_x\right) (\frac{\partial \Phi}{\partial x})
\]
Energy conservation

Transport theorem: If $W_t$ is a region moving with the fluid, then

$$\frac{d}{dt} \int_{W_t} \rho f \, dV = \int_{W_t} \rho \frac{Df}{Dt} \, dV$$

$$\frac{D}{Dt} f = \frac{\partial}{\partial t} f + u \cdot \nabla f$$

$\mathbf{f}(x,t)$ a function

$$\frac{d}{dt} E_{\text{kinetic}} = \frac{d}{dt} \left[ \frac{1}{2} \int_{W_t} \rho \| \mathbf{u} \|^2 \, dV \right] = \frac{1}{2} \int_{W_t} \rho \frac{D\| \mathbf{u} \|^2}{Dt} \, dV$$

$$= \int_{W_t} \rho (\mathbf{u} \cdot \frac{D\mathbf{u}}{Dt}) \, dV = \int_{W_t} - (\mathbf{u} \cdot \nabla (p + \rho \eta)) \, dV$$

$$= \int_{W_t} - \nabla \left[ (p + \rho \eta) \mathbf{u} \right] \, dV = \int_{\partial W_t} (p + \rho \eta) \mathbf{u} \cdot \mathbf{n} \, ds$$

$$= \int_0^{2\pi} \rho g \eta \sqrt{1 + \nabla^2} \frac{\partial \phi}{\partial n} \, dx$$

$$= \int_0^{2\pi} \rho g \eta \eta \, dx = \frac{d}{dt} \frac{1}{2} \int_0^{2\pi} \rho g \eta^2 \, dx$$

$$= \text{potential energy of fluid}$$

Also,

$$\frac{1}{2} \int_{W_t} \rho \| \mathbf{u} \|^2 \, dV = \frac{1}{2} \int_{W_t} \rho \nabla \phi \cdot \nabla \phi \, dV$$

$$= \frac{1}{2} \int_{W_t} \rho \nabla \cdot (\phi \nabla \phi) \, dV = \frac{1}{2} \int_{W_t} \phi \frac{\partial \phi}{\partial n} \, ds$$

So

$$E = \frac{1}{2} \int_0^{2\pi} \rho \phi \frac{\partial \phi}{\partial n} \, ds + \frac{1}{2} \int_0^{2\pi} \rho g \eta^2 \, dx$$

is conserved

Good to monitor this to make sure solution is well resolved.

Also note that

$$\frac{d}{dt} \int_0^{2\pi} \eta \, dx = \int_0^{2\pi} \eta_t \, dx = \int_0^{2\pi} \frac{\partial \phi}{\partial n} \, ds = \int_{W_t} \nabla \phi \, dV = 0$$
For traveling waves, we assume:

\[ \begin{align*}
\eta(x,t) &= \eta(x-ct,0) \\
\phi(x,y,t) &= \phi(x-ct,y,0)
\end{align*} \]

\[ \frac{\partial}{\partial t} \eta(x,t) = -c \eta_x(x-ct,0) \]

\[ \frac{\partial}{\partial t} \phi(x,y,t) = -c \phi_x(x-ct,y,0) \]

\[ \eta_t = \frac{\partial}{\partial t} \eta = \phi_y - \eta_x \phi_x \quad \Rightarrow \quad (\phi_x - c) \eta_x = \phi_y \]

\[ \phi_x + \frac{1}{2} \eta_x \eta_x^2 + g \eta = C(t) \quad \Rightarrow \quad \left( \frac{1}{2} \phi_x - c \right) \phi_x + \frac{1}{2} \phi_y^2 + g \eta = 0 \]

LHS is independent of \( t \), so \( C(t) = \text{const.} \)

In fact, \( C(t) \) may be taken to be zero by choosing the mean value of \( \eta \) correctly. (In numerical simulations, I find that setting \( \int \eta \, dx = 0 \) causes \( C(t) \) to be zero, but I'm not sure how to prove that this is always the case.)

As shown earlier,

\[ \left( \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} \right) = \frac{1}{1 + \eta_x^2} \left( 1 - \eta_x \right) \left( \eta_x \right) \left( \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial y} \right) \]

so everything boils down to computing \( g \phi = \sqrt{1 + \eta_x^2} \frac{\partial \phi}{\partial n} \)

For this, we use a double-layer potential, but now on a \( 2\pi \)-periodic domain.
In the past, we put \( \Omega \) to the left of the curve, but here it's more natural to parametrize \( \Sigma \) so \( \Omega \) is to the right. Rather than think of this as an exterior problem, we'll insert a minus sign in front of \( \Sigma' \):\

\[
\phi(z) = \text{Re} \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{-\Sigma'(\alpha)}{\Sigma(\alpha - z)} \mu(\alpha) \, d\alpha \right\}
\]

we can eliminate the principal value integral by summing over periodic images:

\[
\sum_k \frac{1}{(\Sigma(\alpha + 2\pi k) - z)} = \frac{1}{2} \cot \left( \frac{\Sigma(\alpha) - z}{2} \right)
\]

\[
\phi(z) = \text{Re} \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Sigma'(\alpha)}{2} \cot \left( \frac{z - \Sigma(\alpha)}{2} \right) \mu(\alpha) \, d\alpha \right\}
\]

the quantity inside braces is the complex velocity potential, \( w = \phi + i\psi \), \( \phi = \) scalar velocity potential, \( \psi = \) stream function
The Plimelj formula becomes

\[
W(5(\alpha)^+) = \frac{1}{2} M(\alpha) + \frac{1}{2\pi i} \int_0^{2\pi} \frac{5'(\beta) \cot \left( \frac{5(\alpha) - 5(\beta)}{2} \right)}{2} \mu(\beta) d\beta
\]

We can regularize the integrand as usual with a Hilbert transform:

\[
W(5(\alpha)^+) = \frac{1}{2} M(\alpha) + \frac{1}{2\pi i} \int_0^{2\pi} \left[ \frac{5'(\beta) \cot \left( \frac{5(\alpha) - 5(\beta)}{2} \right)}{2} - \frac{1}{2} \cot \left( \frac{\alpha - \beta}{2} \right) \right] \mu(\beta) d\beta
\]

\[+ \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{2} \cot \left( \frac{\alpha - \beta}{2} \right) \mu(\beta) d\beta \]

\[= \frac{i}{2} H\mu(\alpha) \]

We solve for \( \mu \) by setting \( \text{Re}\{W(5(\alpha)^+)^+\} = \Phi \)

\[
\frac{1}{2} \mu(\alpha) + \frac{1}{2\pi i} \int_0^{2\pi} K(\alpha, \beta) \mu(\beta) d\beta = \Phi(\alpha)
\]

\[K(\alpha, \beta) = \begin{cases} 
\text{Im} \left\{ \frac{5'(\beta) \cot \left( \frac{5(\alpha) - 5(\beta)}{2} \right)}{2} \right\} & \alpha \neq \beta \\
-\text{Im} \left\{ \frac{5''(\alpha)}{2} \frac{5'(\alpha)}{5(\alpha)} \right\} & \alpha = \beta 
\end{cases}
\]
Next we want to compute the velocity.

Cauchy–Riemann: \( \phi_y = -4_x \)

\[
U - iV = \phi_x - i\phi_y = \phi_x + i\phi_y = W_z
\]

\[
= \frac{1}{2\pi i} \int_0^{2\pi} \left[ \frac{\partial}{\partial z} \frac{5'(\alpha)}{2} \cot \left( \frac{z - 5(\alpha)}{2} \right) \right] \mu(\alpha) \, d\alpha
\]

\[
= \frac{1}{2\pi i} \int_0^{2\pi} \left[ -\frac{1}{\partial \alpha} \frac{1}{2} \cot \left( \frac{z - 5(\alpha)}{2} \right) \right] \mu(\alpha) \, d\alpha
\]

\[
= \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{2} \cot \left( \frac{z - 5(\alpha)}{2} \right) \mu'(\alpha) \, d\alpha
\]

\[
\gamma(\alpha) \leftarrow \text{vortex sheet strength}
\]

we get \( G \Phi = \sqrt{1 + \eta_x^2} \frac{\partial \phi}{\partial n} \) from

\[
\hat{a} = \left( \eta_x, 1 \right) \cdot \left( \frac{-\eta_x}{\sqrt{1 + \eta_x^2}} \right) = -\text{Im} \left\{ \frac{(U - iV)(1 + i\eta_x)}{|1 + i\eta_x|} \right\}
\]

\[
15'(\alpha) \left| \frac{\partial \phi}{\partial n} (z, \alpha) \right| = -\text{Im} \left\{ \frac{1}{2\pi i} \int_0^{2\pi} \frac{5'(\alpha)}{2} \cot \left( \frac{z - 5(\beta)}{2} \right) \delta(\beta) \, d\beta \right\}
\]

Taking the limit as \( z \to 5(\alpha)^+ \), we get

\[
15'(\alpha) \left| \frac{\partial \phi}{\partial n} (5(\alpha)^+) \right| = -\text{Im} \left\{ \frac{1}{2} \gamma(\alpha) + \frac{1}{2\pi i} \int_0^{2\pi} \frac{5'(\alpha)}{2} \cot \left( \frac{5(\alpha) - 5(\beta)}{2} \right) \gamma(\beta) \, d\beta \right\}
\]

\( G \Phi(\alpha) \)
Finally, we regularize the integrand (and use \(-\text{Im}(A) = \text{Re}(iA)\))

\[
\chi \Phi(a) = \text{Re}\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{\Phi'(a)}{2} \cot\left(\frac{\Phi(a) - \Phi(\beta)}{2}\right) - i \frac{1}{2} \cot\left(\frac{a - \beta}{2}\right)\right] \chi(\beta) \, d\beta \right. \\
+ \left. \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \cot\left(\frac{a - \beta}{2}\right) \chi(\beta) \, d\beta \right\}
\]

\[
\chi \Phi(a) = \frac{1}{2} \mathcal{H} \Phi(a) + \frac{1}{2\pi} \int_0^{2\pi} \alpha(\beta, \beta) \chi(\beta) \, d\beta
\]

\[
\alpha(\alpha, \beta) = \left\{ \begin{array}{ll}
\text{Re}\left\{ \frac{\Phi'(a)}{2} \cot\left(\frac{\Phi(a) - \Phi(\beta)}{2}\right) - i \frac{1}{2} \cot\left(\frac{a - \beta}{2}\right)\right\} & \alpha \neq \beta \\
\text{Re}\left\{ \frac{\Phi''(a)}{2 \Phi'(a)} \right\} & \alpha = \beta
\end{array} \right.
\]

To compute the pressure, we go back to Bernoulli:

\[
\phi_t + \frac{1}{2} \|
\nabla \phi \|^2 + \frac{\rho}{
} + g y = C(t)
\]

In the traveling case, we had \(C(t) = 0\) and \(\phi_t = -c \phi_x\)

so

\[
\rho(z) = \rho \left( c \phi_x - \frac{1}{2} \phi_x^2 - \frac{1}{2} \phi_y^2 - g y \right)
\]

where we use the formula \(u - i v = \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{2} \cot\left(\frac{z - \Phi(\alpha)}{2}\right) \chi(\alpha) \, d\alpha\)

To evaluate \(\phi_x = u, \phi_y = v\) at each point \(z\) of the mesh.