1. Consider the scheme \(D_t^+ U = D_x^+ D_x^- U + 10 D_x^0 U\)

for the equation \(\begin{cases} U_t = U_{xx} + 10 U_x \\ U(x,0) = g(x) \end{cases}\) \(-\infty < x < \infty\)

(a) write down the operator \(B\) such that \(U^{n+1} = B U^n\)

(b) show that \(|B|_{\text{inf}} = 1\) if \(y = \frac{h}{\Delta x} \leq \frac{1}{2}\) and \(h \leq \frac{1}{4}\)

(c) show that \(|B|_{\text{inf}} = 1\) if \(y \leq \frac{1}{12}\) and \(h \leq \frac{1}{15}\)

(d) compute the amplification factor \(G(\xi)\) for this scheme and plot it for \(y = \frac{1}{12}, h = \frac{1}{12}\)

2. Use the above scheme to solve the equation on the finite interval \(0 \leq x \leq 1\) with periodic boundary conditions \(U(1) = U(0), U_x(1) = U_x(0)\) and initial conditions \(g(x) = \sin(2\pi x)\) up to time \(T = 0.1\)

Compute the order of convergence with \(y = \frac{1}{12}\) and \(y = \frac{1}{16}\).

3. A circulant matrix is constant along diagonals with entries that "wrap around". For convenience, let's index our matrices starting at zero. Show that \(A U = U A\)

\[
A_{k\ell} = \begin{cases} \hat{r}_{k-\ell} & k \geq \ell \\ \hat{r}_{N+k-\ell} & k < \ell \end{cases}
\]

Example \((N=4)\): 

\[
\Lambda = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}
\]