2.16 Polar co-ordinates

One-dimensional problems often result from physical systems in three dimensions which have cylindrical or spherical symmetry. In polar co-ordinates the simple heat equation becomes

$$\frac{\partial u}{\partial t} = \frac{1}{r^\alpha \frac{\partial}{\partial r} \left( r^\alpha \frac{\partial u}{\partial r} \right)}$$

(2.156)

or

$$u_t = u_{rr} + \frac{\alpha}{r} u_r,$$

(2.157)

where $\alpha = 0$ reduces to the case of plane symmetry which we have considered so far, while $\alpha = 1$ corresponds to cylindrical symmetry and $\alpha = 2$ to spherical symmetry. The methods just described can easily be applied to this equation, either in the form (2.156), or in the form (2.157). Examination of the stability restrictions in the two cases shows that there is not much to choose between them in this particular situation. However, in each case there is clearly a problem at the origin $r = 0$. 

The right-hand side of the equation, giving the explicit difference scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{1}{(\Delta x)^2} \left[ p_j^{n+1/2} (U_j^{n+1} - U_{j+1}^n) - p_j^{n-1/2} (U_j^n - U_{j-1}^n) \right],$$

(2.153)

which gives in explicit form

$$U_j^{n+1} = \left(1 - \nu (p_j^{n+1/2} + p_j^{n-1/2})\right) U_j^n + \nu p_j^{n+1/2} U_{j+1}^n + \nu p_j^{n-1/2} U_{j-1}^n,$$

(2.154)

This shows that the form of error analysis which we have used before will again apply here, with each of the coefficients on the right-hand side being non-negative provided that

$$\nu P \leq \frac{1}{2},$$

(2.155)

where $P$ is an upper bound for the function $p(x,t)$ in the region. So this scheme gives just the sort of time step restriction which we should expect, without any restriction on the size of $\Delta x$.

The same type of difference approximation can be applied to give an obvious generalisation of the $\theta$-method. The details are left as an exercise, as is the calculation of the leading terms of the truncation error (see Exercises 7 and 8).
Parabolic equations in one space variable

A consideration of the symmetry of the solution, in either two or three dimensions, shows that \( \partial u / \partial r = 0 \) at the origin; alternatively, (2.157) shows that either \( u_{rr} \) or \( u_t \), or both, would be infinite at \( r = 0 \), were \( u_r \) non-zero. Now keep \( t \) constant, treating \( u \) as a function of \( r \) only, and expand in a Taylor series around \( r = 0 \), giving

\[
    u(r) = u(0) + ru_r(0) + \frac{1}{2} r^2 u_{rr}(0) + \ldots \\
    = u(0) + \frac{1}{2} r^2 u_{rr}(0) + \ldots \quad (2.158)
\]

and

\[
    \frac{1}{r^\alpha \partial r} \left( r^\alpha \frac{\partial u}{\partial r} \right) = \frac{1}{r^\alpha \partial r} \left[ r^\alpha u_r(0) + r^{\alpha+1} u_{rr}(0) + \ldots \right] \\
    = \frac{1}{r^\alpha} \left[ (\alpha + 1) r^\alpha u_{rr}(0) + \ldots \right] \\
    = (\alpha + 1) u_{rr}(0) + \ldots \quad (2.159)
\]

Writing \( \Delta r \) for \( r \) in (2.158) we get

\[
    u(\Delta r) - u(0) = \frac{1}{2}(\Delta r)^2 u_{rr}(0) + \ldots \quad (2.160)
\]

and we thus obtain a difference approximation to be used at the left end of the domain,

\[
    \frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{2(\alpha + 1)}{(\Delta r)^2} \left( U_j^n - U_0^n \right). \quad (2.161)
\]

This would also allow any of the \( \theta \)-methods to be applied.

An alternative, more physical, viewpoint springs directly from the form (2.156). Consider the heat balance for an annular region between two surfaces at \( r = r_{j-1/2} \) and \( r = r_{j+1/2} \) as in Fig. 2.11: the term \( r^\alpha \partial u / \partial r \) on the right-hand side of (2.156) is proportional to a heat flux times a surface area; and the difference between the fluxes at surfaces with radii \( r_{j-1/2} \) and \( r_{j+1/2} \) is applied to raising the temperature in a volume which is proportional to \( (r_{j+1/2}^{\alpha+1} - r_{j-1/2}^{\alpha+1}) / (\alpha + 1) \). Thus on a uniform mesh of spacing \( \Delta r \), a direct differencing of the right-hand side of (2.156) gives

\[
    \frac{\partial U_j}{\partial t} \approx \frac{\alpha + 1}{r_{j+1/2}^{\alpha+1} - r_{j-1/2}^{\alpha+1}} \delta_r \left( r_j^{\alpha+1} \delta_r U_j \right) \\
    = \frac{(\alpha + 1) \left[ r_{j+1/2}^{\alpha+1} U_{j+1} - \left( r_{j+1/2}^{\alpha} + r_{j-1/2}^{\alpha} \right) U_j + r_{j-1/2}^{\alpha+1} U_{j-1} \right]}{\left[ r_{j+1/2}^{\alpha+1} + r_{j+1/2}^{\alpha} + r_{j-1/2}^{\alpha+1} + \ldots + r_{j-1/2}^{\alpha} \right] (\Delta r)^2} \\
    \text{for } j = 1, 2, \ldots \quad (2.162a)
\]

Fig. 2.11. Polar co-ordinates.

At the origin where there is only one surface (a cylinder of radius \( r_{1/2} = \frac{1}{2} \Delta r \) when \( \alpha = 1 \), a sphere of radius \( r_{1/2} \) when \( \alpha = 2 \)) one has immediately

\[
    \frac{\partial U_0}{\partial t} \approx \frac{\alpha + 1}{r_{1/2}^{\alpha+1}} \frac{\delta_r \left( U_{1/2} - U_0 \right)}{\Delta r} = 2(\alpha + 1) \frac{U_1 - U_0}{(\Delta r)^2}, \quad (2.162b)
\]

which is in agreement with (2.161). Note also that (2.162a) is identical with difference schemes obtained from either (2.156) or (2.157) in the case of cylindrical symmetry \( (\alpha = 1) \); but there is a difference in the spherical case because \( r_{j+1/2}^{\alpha+1} + r_{j+1/2}^{\alpha} r_{j-1/2}^{\alpha+1} + r_{j-1/2}^{\alpha+1} \) is not the same as \( 3r_{j}^{\alpha+1} \).

The form (2.162a) and (2.162b) is simplest for considering the condition that a maximum principle should hold. From calculating the coefficient of \( U_j^n \) in the \( \theta \)-method, one readily deduces that the worst case occurs at the origin and leads to the condition

\[
    2(\alpha + 1)(1 - \theta) \Delta t \leq (\Delta r)^2. \quad (2.162c)
\]

This becomes more restrictive as the number of space dimensions increases in a way which is consistent with what we shall see in Chapter 3.