(1) (Chain rule). Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \), \( g : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) and \( u : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be given by

\[
\begin{align*}
f(y) &= \begin{pmatrix} y_1 - e^{y_2} \\ y_2 \\ 2(y_1 - 2)^3 \end{pmatrix}, & g(x) &= \begin{pmatrix} x_1 + 2x_2 + 3x_3 \\ \ln(x_1 + 2x_2) \end{pmatrix}, & u(x) &= f(g(x)).
\end{align*}
\]

Let \( x_0 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \) and \( y_0 = g(x_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). Write out the formula for \( u(x) \) in terms of \( x \), compute the Jacobians \( Df(y_0), Dg(x_0) \) and \( Du(x_0) \), and verify that \( Du(x_0) = Df(g(x_0)) \cdot Dg(x_0) \).

(2) In class we studied the behavior of Euler’s method for the equation

\[
\begin{align*}
y' &= f(y), \\
y(0) &= 0, & f(y) &= 1 + |y - 1|
\end{align*}
\]

and showed that the numerical solution \( y_K \) at \( t_K = 1 \) satisfies

\[
y_K = \frac{e}{2} + \frac{e}{2} \left( \ln 2 - \frac{1}{2} \right) h + \frac{e}{2} \left[ \theta(1 - \theta) + \frac{(\ln 2)^2}{2} - \ln 2 + \frac{11}{24} \right] h^2 + O(h^3),
\]

where \( e = \exp(1) \) and \( \theta \) is a more or less random parameter between 0 and 1 that depends on where the numerical solution crosses \( y = 1 \). Thus, in spite of the discontinuity in \( \frac{\partial f}{\partial y} \), it appears that there is a nice function \( \varepsilon(t) \) such that

\[
y_n = y(t_n) + h\varepsilon(t_n) + O(h^2), \quad 0 \leq n \leq K = h^{-1}.
\]

Solve the variational equation

\[
\varepsilon'(t) = A(t)\varepsilon(t) + b(t), \quad A(t) = f'(y(t)), \quad b(t) = -\frac{1}{2}y''(t)
\]

by hand (i.e. analytically) to verify that \( \varepsilon(1) = \frac{e}{2} \left( \ln 2 - \frac{1}{2} \right) \). Hint: solve up to \( t = \ln 2 \) and use the result as the initial condition for the new equation.
(3) Derive the four step Adams-Bashforth method and use it to solve the problems

(a) (2-d gravity) $x' = u, \ y' = v, \ u' = -x/(x^2 + y^2)^{3/2}, \ v' = -y/(x^2 + y^2)^{3/2},$

$x(0) = 1, \ y(0) = 0, \ u(0) = 0, \ v(0) = 1, \ T = 2\pi$

(b) (non-smooth $f$) $y' = 1 + |y - 1|, \ y(0) = 0, \ T = 1$

Use the exact solutions to start the method going. Compute the error $\|e_K\| = \|y_K - y(t_K)\|$ at $t_K = T$ with $K = 12, 15, 19, 24, 30, 38, 48, 60, 76, \ldots, 6144, 7680, 9728$ using the exact solution for $y(T)$. (Note the pattern: $x_{k+2} = 2x_k$). Plot $\log_{10}(\|e_K\|)$ vs. $\log(h)$ (where $h = T/K$) and do a linear regression to find the order at which the numerical solution is converging to the exact solution.