Theorem 3 page 449

For a given power series \( \sum_{n=0}^{\infty} C_n (x-a)^n \), there are only three possibilities:

(i) the series converges only when \( x=a \). \( R=0 \)
(ii) the series converges for all \( x \). \( R=\infty \)
(iii) there is a positive number \( R \) such that
   (a) the series converges for \( |x-a|<R \)
   (b) it diverges for \( |x-a|>R \)

proof: Let \( R \) be the least upper bound of the set of all \( r \geq 0 \) such that the sequence \( |C_n| r^n \to 0 \) as \( n \to \infty \).

This set is non-empty since \( r=0 \) satisfies \( |C_n| r^n \to 0 \).
If the set is unbounded, i.e. if there are arbitrarily large values of \( r \) such that \( |C_n| r^n \to 0 \) as \( n \to \infty \), then we define \( R=\infty \). Otherwise, the completeness axiom (see page 418) guarantees that \( R \) exists and is finite.

cases (ii) and (iii a): suppose \( |x-a|<R \) \( (\text{guaranteed in case } ii) \) where \( R=\infty \)

- must show \( \sum_{n=0}^{\infty} C_n (x-a)^n \) converges

- since \( |x-a| \) is not an upper bound for the set \( (R \text{ is the least upper bound}) \) there is an \( r>|x-a| \) such that \( |C_n| r^n \to 0 \) as \( n \to \infty \).
• Convergent sequences are bounded, so there is a constant $C$ such that $|c_n| r^n \leq C$ for all $n$.

$$\sum_{n=0}^{\infty} |c_n(x-a)^n| = \sum_{n=0}^{\infty} |c_n| r^n \frac{|x-a|^n}{r^n} \leq C \sum_{n=0}^{\infty} \left( \frac{|x-a|}{r} \right)^n$$

Comparison test

• Series is absolutely convergent, hence convergent.

Cases (i) and (iii): Suppose $|x-a| > R$

• Must show $\sum_{n=0}^{\infty} c_n(x-a)^n$ diverges.

• $|x-a|$ does not belong to the set as $R$ is an upper bound for the elements in the set.

• So $|c_n||x-a|^n$ does not converge to zero.

• Neither does $c_n(x-a)^n$ (if it did, so would $|c_n||x-a|^n$).

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$ diverges.