Sample Final Exam 2

You are allowed one 8.5×11 sheet of notes with writing on both sides. This sheet must be turned in with your exam. *Calculators are not allowed.*

0. (1 point) write your name, section number, and GSI's name on your exam.

1. (3 points) give precise definitions of the following statements or expressions:
(a) $f(x)$ is neither even nor odd
(b) $\int f(x)\,dx$
(c) $\int_a^b f(x)\,dx$

Solution:
(a) There exist numbers x_1 and x_2 such that $f(-x_1) \neq f(x_1)$ and $f(-x_2) \neq -f(x_2)$.
(b) $\int f(x)\,dx$ is any antiderivative of $f(x)$, i.e. a function $F(x)$ such that $F'(x) = f(x)$.
(c) the definite integral is defined as

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i,$$

where the limit is over all partitions $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of the interval $[a, b]$ into subintervals of length $\Delta_i = x_i - x_{i-1}$, and x_i^* is a sample point in the ith interval $[x_{i-1}, x_i]$.

2. (4 points) Show that the tangent lines to the curves $y = x^3$ and $x^2 + 3y^2 = 1$ are perpendicular where the curves intersect.

Solution:
The slope of the tangent line of the first curve is $m_1 = y' = 3x^2$.
For the second, differentiate implicitly:

$$2x + 6yy' = 0 \qquad \Rightarrow \qquad y' = -\frac{x}{3y}$$

When the curves intersect, we have $y = x^3$, so $m_2 = y' = -1/(3x^2)$. Since $m_2 = -1/m_1$, these tangent lines are perpendicular.

3. (3 points) Evaluate $\displaystyle\int_0^1 \frac{\tan^{-1} x}{1+x^2}\, dx$.

Solution:
Let $u = \tan^{-1} x$. Then $du = \dfrac{dx}{1+x^2}$ and the limits of integration become

$$x = 0 \quad \to \quad u = 0, \qquad x = 1 \quad \to \quad u = \frac{\pi}{4}.$$

(The latter condition comes from solving $\tan u = \frac{\sin u}{\cos u} = 1$ for u). So

$$\int_0^1 \frac{\tan^{-1} x}{1+x^2}\, dx = \int_0^{\pi/4} u\, du = \frac{u^2}{2}\Big|_0^{\pi/4} = \frac{\pi^2}{32}.$$

4. If f is continuous and $\int_0^4 f(x)\, dx = 6$, find $\int_0^2 f(2x)\, dx$.

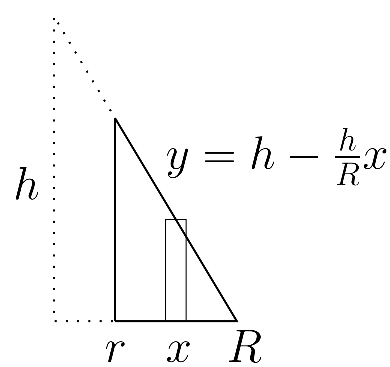
Solution: Let $u = 2x$. Then $du = 2dx$ so

$$\int_0^2 f(2x)\, dx = \int_0^4 f(u) \left(\frac{du}{2}\right) = \frac{1}{2}\int_0^4 f(u)\, du = 3.$$

5. (5 points) A right circular cone of height h and base radius R has a hole of radius r drilled through its center (from the tip to the center of the base). Find the volume of the solid that remains.

Solution:

$$\begin{aligned} V &= \int_r^R 2\pi x \left(h - \frac{h}{R}x\right) dx \\ &= 2\pi h \left\{ \frac{x^2}{2}\Big|_r^R - \frac{x^3}{3R}\Big|_r^R \right\} \\ &= 2\pi h \left(\frac{R^2}{2} - \frac{r^2}{2} - \frac{R^3}{3R} + \frac{r^3}{3R} \right) \\ &= \pi h \left(\frac{R^2}{3} - r^2 + \frac{2r^3}{3R} \right). \end{aligned}$$



6. (5 points) Let $f(x) = \tanh^{-1}(\sin x)$ and $g(x) = \ln|\sec x + \tan x|$. Compute $f'(x)$, $g'(x)$, $f(n\pi)$ and $g(n\pi)$ with n an integer. What do you conclude?

Solution: $\frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2}$ and $\frac{d}{dx}\ln|x| = \frac{1}{2}\frac{d}{dx}\ln x^2 = \frac{1}{2}\cdot\frac{2x}{x^2} = \frac{1}{x}$. So by the chain rule:

$$f'(x) = \frac{\cos x}{1-\sin^2 x} = \sec x, \qquad g'(x) = \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x} = \sec x,$$
$$f(n\pi) = \tanh^{-1}(\sin n\pi) = \tanh^{-1}(0) = 0,$$
$$g(n\pi) = \ln|\sec(n\pi) + \tan(n\pi)| = \ln|\pm 1 + 0| = \ln(1) = 0.$$

Consider one of the intervals (a, b) on which f and g are defined (i.e. let $a = (n - \frac{1}{2})\pi$ and $b = (n + \frac{1}{2})\pi$ for some integer n). Since $f'(x) = g'(x)$ on this interval, the mean value theorem implies that there is a constant C such that $f(x) - g(x) = C$. Since $f(n\pi) = g(n\pi)$, $C = 0$. So $f(x) = g(x)$ for all x in their domains.

7. (5 points) A boat leaves a dock at 2:00 PM and travels due south at a speed of 20 km/h. Another boat has been heading due east at 10 km/h and reaches the same dock at 3:00 PM. At what time were the two boats closest together? Verify that the distance was minimized using one of the derivative tests.

Solution:
Let y be the distance from the first boat to the dock.
Let x be the distance from the second boat to the dock.
Let s be the distance from the first boat to the second boat.
Let t be the time (in hours) since 2:00 PM. Then

$$y = 20t, \qquad x = 10 - 10t, \qquad s^2 = x^2 + y^2,$$
$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2(10-10t)(-10) + 2(20t)(20) = 0 \text{ at a minimum.}$$

Now we divide by 200 and solve for t:

$$(1-t)(-1) + (2t)(2) = -1 + 5t = 0, \qquad t = 1/5.$$

The first derivative test is easier here since the sign of $\frac{ds}{dt}$ is the same as that of $s\frac{ds}{dt} = 200(-1+5t)$, which changes from negative to positive as t crosses $1/5$, indicating that s achieves a minimum at $t = 1/5$.

8. (6 points) Let $f(x) = \dfrac{x^2(\sqrt{x^2+3} - x - 1)}{x^2 - 1}$.

(a) find all vertical and horizontal asymptotes of f.

(b) show that $y = -2x - 1$ is a slant asymptote, i.e. $\lim_{x \to -\infty} [f(x) - (-2x - 1)] = 0$.

Hint for (b): first show that $\lim_{x \to -\infty}[\sqrt{x^2+3} + x] = 0$, *then manipulate* $[f(x) + 2x + 1]$ *to make use of this.*

(a) vertical asymptotes: candidates are $x=1$, $x=-1$

at $x=1$, numerator is $(1)(\sqrt{4}-1-1) = 0$. $\frac{0}{0}$ form. use l'Hospital.

$$\lim_{x\to 1} f(x) = \lim_{x \to 1} \frac{2x(\sqrt{x^2+3} - x - 1) + x^2\left(\frac{2x}{2\sqrt{x^2+3}} - 1\right)}{2x}$$

$$= \frac{2(0) + (1)\left(\frac{1}{2} - 1\right)}{2} = -\frac{1}{4} \neq \infty$$

so f does not have a vertical asymptote at $x=1$.

at $x=-1$, numerator is $(1)(\sqrt{4}+1-1) = 2$

denominator is $(x-1)(x+1) = (-2)(0) = 0$

$$\lim_{x \to -1^+} f(x) = \frac{2}{(-2)(0^+)} = -\infty$$

$$\lim_{x \to -1^-} f(x) = \frac{2}{(-2)(0^-)} = +\infty$$

f has a vertical asymptote at $x=-1$.

horizontal asymptotes:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{(\sqrt{x^2+3})^2 - \overbrace{(x+1)^2}^{x^2+2x+1}}{(1 - \frac{1}{x^2})(\sqrt{x^2+3} + x + 1)} = \lim_{x \to \infty} \frac{2 - 2x}{(1 - \frac{1}{x^2})(\sqrt{x^2+3} + x + 1)}$$

$$= \lim_{x \to \infty} \frac{(2/x) - 2}{(1 - \frac{1}{x^2})(\sqrt{1 + \frac{3}{x^2}} + 1 + \frac{1}{x})} = \frac{0 - 2}{(1-0)(\sqrt{1+0} + 1 + 0)} = \frac{-2}{2} = -1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\sqrt{x^2+3}-x-1}{1-1/x^2} = \frac{+\infty}{1-0} = \infty$$

So $\boxed{y=-1 \text{ is the only horizontal asymptote}}$

(b) $$\lim_{x\to-\infty} \sqrt{x^2+3}+x = \lim_{x\to-\infty} \frac{(x^2+3)-x^2}{\sqrt{x^2+3}-x} = \lim_{x\to-\infty} \frac{3/|x|}{\sqrt{1+\frac{3}{x^2}} - \frac{x}{|x|}}$$

$$= \lim_{x\to-\infty} \frac{3/|x|}{\sqrt{1+\frac{3}{x^2}}+1} = \frac{0}{2} = 0$$

x is negative so $|x| = -x$ and $\frac{x}{|x|} = -1$

$$\lim_{x\to-\infty} \left[\frac{x^2\left(\sqrt{x^2+3}-x-1\right)}{x^2-1} - (-2x-1) \right]$$

$$= \lim_{x\to-\infty} \left[\frac{\left(\sqrt{x^2+3}+x\right)-2x-1}{1-1/x^2} + 2x+1 \right]$$

$$= \lim_{x\to-\infty} \left[\frac{\left(\sqrt{x^2+3}+x\right) \overbrace{-2x-1+2x+1}^{0} - \frac{2}{x} - \frac{1}{x^2}}{1-1/x^2} \right]$$

$$= \frac{0-0-0-0}{1-0} = 0$$

9. (9 points) A model rocket is fired vertically upward from rest. Its acceleration (in m/s^2) for the first two seconds is $a(t) = 24t$, at which time the fuel is exhausted and it becomes a freely "falling" body (with constant acceleration $a(t) = -8\,m/s^2$; the earth's gravity was unusually weak that day.) 10 seconds later, the parachute opens and the velocity v (which is negative at this point) slows according to the differential equation

$$\frac{dv}{dt} = -(v - v_s), \qquad v_s = -5\,m/s^2 \tag{1}$$

until it hits the ground.
(a) Determine the position $s(t)$, velocity $v(t)$, and acceleration $a(t)$ for $0 \le t \le 12$. (The parachute opens at $t = 12$).
(b) At what time does the rocket reach its maximum height, and what is that height?
(c) Find $v(t)$ for $t \in [12, T]$, where T is the time when the rocket hits the ground. (you don't have to compute T, which turns out to be very close to 29).
(d) sketch the graphs of $a(t)$, $v(t)$ and $s(t)$ from $0 \le t \le T$. Be sure your curves are qualitatively correct even though you did not work out the formulas for $s(t)$ or $a(t)$ for $t > 12$.

(a) $0 \le t \le 2$:

$$a(t) = 24t$$
$$v(t) = v(0) + \int_0^t a(u)\,du = 0 + 12u^2\Big|_0^t = 12t^2$$
$$s(t) = s(0) + \int_0^t v(u)\,du = 0 + 4u^3\Big|_0^t = 4t^3$$
$$v(2) = 48$$
$$s(2) = 32$$

$2 \le t \le 12$:

$$a(t) = -8$$
$$v(t) = v(2) + \int_2^t (u)\,du = 48 - 8u\Big|_2^t = 64 - 8t$$
$$s(t) = s(2) + \int_2^t v(u)\,du = 32 + 64(t-2) - 4(t^2-4)$$
$$= -80 + 64t - 4t^2$$

$$v(12) = \overset{8(8)}{64} - 8(12) = 8(-4) = -32$$

$$s(12) = -80 + \underset{768}{64(12)} - \underset{576}{4(144)} = 112$$

(b) max height: $v(t_{max}) = 0$

$$64 - 8t_{max} = 0$$

$$t_{max} = 8$$

$$s(t_{max}) = -80 + 64(8) - 4(64) = 176$$

(c) $\frac{dv}{dt} = -(v+5)$, $v(12) = -32$

$$w = v + 5$$

$$\frac{dw}{dt} = \frac{dv}{dt} = -(v+5) = -w$$

$$w(t) = Ce^{-t}$$

$$w(12) = v(12) + 5 = -27 = Ce^{-12} \Rightarrow C = -27e^{12}$$

$$v(t) = w(t) - 5 = -27e^{12}e^{-t} - 5$$

$$\boxed{v(t) = -5 - 27e^{-(t-12)}}$$

check: $v'(t) = 27e^{-(t-12)} = -(v+5)$ ✓

(d)