Sample Final Exam 2

You are allowed one 8.5 × 11 sheet of notes with writing on both sides. This sheet must be turned in with your exam. Calculators are not allowed.

0. (1 point) write your name, section number, and GSI’s name on your exam.

1. (3 points) give precise definitions of the following statements or expressions:
   (a) $f(x)$ is neither even nor odd
   (b) $\int f(x) \, dx$
   (c) $\int_a^b f(x) \, dx$

2. (4 points) Show that the tangent lines to the curves $y = x^3$ and $x^2 + 3y^2 = 1$ are perpendicular where the curves intersect.

3. (3 points) Evaluate $\int_0^1 \tan^{-1} \left( \frac{1}{x} \right) \, dx$.

4. If $f$ is continuous and $\int_0^4 f(x) \, dx = 6$, find $\int_0^2 f(2x) \, dx$.

5. (5 points) A right circular cone of height $h$ and base radius $R$ has a hole of radius $r$ drilled through its center (from the tip to the center of the base). Find the volume of the solid that remains.

6. (5 points) Let $f(x) = \tanh^{-1}(\sin x)$ and $g(x) = \ln |\sec x + \tan x|$. Compute $f'(x)$, $g'(x)$, $f(n\pi)$ and $g(n\pi)$ with $n$ an integer. What do you conclude?

7. (5 points) A boat leaves a dock at 2:00 PM and travels due south at a speed of 20 km/h. Another boat has been heading due east at 10 km/h and reaches the same dock at 3:00 PM. At what time were the two boats closest together? Verify that the distance was minimized using one of the derivative tests.

8. (6 points) Let $f(x) = \frac{x^2(\sqrt{x^2 + 3} - x - 1)}{x^2 - 1}$.
   (a) find all vertical and horizontal asymptotes of $f$.
   (b) show that $y = -2x - 1$ is a slant asymptote, i.e. $\lim_{x \to -\infty} \left[ f(x) - (-2x - 1) \right] = 0$.
   Hint for (b): first show that $\lim_{x \to -\infty} [\sqrt{x^2 + 3} + x] = 0$, then manipulate $[f(x) + 2x + 1]$ to make use of this.

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9. (9 points) A model rocket is fired vertically upward from rest. Its acceleration (in m/s²) for the first two seconds is \( a(t) = 24t \), at which time the fuel is exhausted and it becomes a freely “falling” body (with constant acceleration \( a(t) = -8 \) m/s²; the earth’s gravity was unusually weak that day.) 10 seconds later, the parachute opens and the velocity \( v \) (which is negative at this point) slows according to the differential equation

\[
\frac{dv}{dt} = -(v - v_s), \quad v_s = -5 \text{ m/s}
\]  

until it hits the ground.

(a) Determine the position \( s(t) \), velocity \( v(t) \), and acceleration \( a(t) \) for \( 0 \leq t \leq 12 \). (The parachute opens at \( t = 12 \).)

(b) At what time does the rocket reach its maximum height, and what is that height?

(c) Find \( v(t) \) for \( t \in [12, T] \), where \( T \) is the time when the rocket hits the ground. (you don’t have to compute \( T \), which turns out to be very close to 29).

(d) sketch the graphs of \( a(t) \), \( v(t) \) and \( s(t) \) from \( 0 \leq t \leq T \). Be sure your curves are qualitatively correct even though you did not work out the formulas for \( s(t) \) or \( a(t) \) for \( t > 12 \).