(1a) Show that if $A : V \to V$ is a linear operator on an inner product space $V$ satisfying $A^* = A^{-1}$, then $A$ preserves lengths and inner products. Such matrices are called unitary.

(1b) Let $V = S_N$ with inner product $\langle y, z \rangle = \frac{1}{N} \sum_{j=0}^{N-1} y_j \overline{z}_j$. Show that the adjoint of a matrix in this inner product is its conjugate transpose. Now use the fact that $\frac{1}{\sqrt{N}} F_N \bar{F}_N = I$ and $F_N^T F_N = F_N$ to conclude that $\frac{1}{\sqrt{N}} F_N$ and $\frac{1}{\sqrt{N}} \bar{F}_N$ are both unitary.

(1c) Show that the $2$-norm of a diagonal matrix $D$ acting on $S_N$ is equal to the largest of the absolute values of the diagonal entries, i.e.

$$\sup_{y \neq 0} \frac{\|Dy\|_{S_N}}{\|y\|_{S_N}} = \max_{0 \leq j \leq N-1} |D_{jj}|.$$ 

(1d) Compute the $2$-norm of the operator $Ly = h \ast y$, where $h \in S_N$.

(2) problem 15 page 153. To make your life simpler, assume that matrices are indexed from $0$ to $N - 1$ so that $a_\ell = A_{\ell,0}$, $x_\ell = X_{\ell,0}$, $y_\ell = Y_{\ell,0}$ for $\ell = 0, \ldots, n - 1$. In part (c), in addition to showing that $n^{-1} F_n^T A F_n = D$ is diagonal, show that $n^{-1} F_n^T A F_n = S$ is also diagonal. I consider this second version to be the more natural diagonalization.

(3) problem 16 page 153.

(4) Consider the heat equation on a periodic domain

$$u_t = u_{xx}, \quad u(x, 0) = u_0(x), \quad u(0, t) = u(2\pi, t).$$

(a) Check that $u(x, t) = e^{-t} \cos x + e^{-4t} \sin 2x$ is the solution with initial condition $u_0(x) = \cos x + \sin 2x$. Use this initial condition in (b) and (d) below.

(b) code up the finite difference method $u_{j+1}^n = \nu u_{j-1}^n + (1 - 2\nu) u_j^n + \nu u_{j+1}^n$ with $\nu = \frac{\Delta t}{\Delta x^2}$ using $\Delta t = 1/N$, $\Delta x = 2\pi/M$, $N = 1000$, $M = 100$. Note that periodicity requires that when $j = 0$, $u_{j-1}^n = u_{M-1}^n$ and when $j = M - 1$, $u_{j+1}^n = u_0^n$. Plot the numerical solution $\{u_j^n\}_{j=0}^{M-1}$ at $n = N$ (i.e. $t = 1$). Also plot the error $E_j^N = u_j^N - u(j\Delta x, 1)$.

(c) we can write the above numerical scheme as $u_{n+1}^j = \mathcal{L}(u^n)$, where $Ly = h \ast y$ and $h = (\ldots, 0, \nu, 1 - 2\nu, \nu, 0, \ldots)$. Show that the transfer function of $h$ is $\hat{h}(\phi) = 1 - 4\nu \sin^2(\phi/2)$.

(d) compute the numerical solution in (b) a different way: let

$$v = \text{fft}(u_0), \quad w_k = \hat{h}(\phi_k)^N v_k, \quad 0 \leq k \leq M - 1, \quad U^N = \text{ifft}(w),$$

where $\phi_k = 2\pi k/M$. Plot the real and imaginary parts of the difference $\{U_j^N - u_j^N\}_{j=0}^{M-1}$ to show that the two methods give the same answer (up to roundoff error). Be careful of “off by one” errors in the indices.