DEFINABLE VIZING THEOREMS

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Abstract. In this note, we show that if G is a Borel graph of finite maximum degree, then its Baire measurable edge chromatic number is at most 1 more than its (discrete) edge chromatic number. This bound follows from a more general one stated in terms of asymptotic separation index and which allows for multigraphs. Our arguments are largely based on work of Kierstead, who in [Kie81] used similar arguments to produce recursive colorings of graphs on, say, the natural numbers. In a future paper, we will expand on this note by exploring the connection between these two combinatorial settings more comprehensively.

1. Introduction

A theorem of Vizing states that if G is a multigraph with finite maximum degree Δ and finite maximum edge multiplicity p, then $\chi'(G) \leq \Delta + p$. In this note, we are interested in bounding $\chi' - \Delta$ in the descriptive setting. Our main theorem gives such a bound in terms of the parameter $\mathrm{asi}(G)$. This is the *asymptotic separation index*, defined in [CJM$^+$20].

Theorem 1.1. *Let G be a Borel multigraph with maximum degree $\Delta \in \omega$ and maximum edge multiplicity $p \in \omega$. Then*

$$\chi'_B(G) \leq \chi'(G) + p \cdot \mathrm{asi}(G) \leq \Delta + p(\mathrm{asi}(G) + 1).$$

Note that the second inequality is just the classical Vizing theorem. If $\mathrm{asi}(G)$ is not finite, the Theorem simply asserts a countable Borel edge coloring of G, which exists by local finiteness [KST99]. Therefore assume $\mathrm{asi}(G) \in \omega$.

In the case $p = 1$, this theorem generalizes the main result from [BW21], which proved the same bound for bipartite graphs. The overall proof structure of the two results is also identical. The bound in that result holds regardless of p, however, whereas the bound of Theorem 1.1 becomes unoptimal for bipartite multigraphs when $p > 1$.

This theorem and its proof, especially in the case $\mathrm{asi}(G) = 1, p = 1$, are heavily inspired by Theorem 4.1 from [Kie81]. That theorem gave a similar bound in a setting which might be called "recursive graph combinatorics": One is given a countable graph which is in some sense computable, and is asked to produce a similarly computable solution to some combinatorial problem, e.g. an edge coloring. Kierstead's techniques throughout that paper, and related techniques from related papers, have a flavor similar to many common proof structures in descriptive combinatorics, and in a future paper we will more comprehensively explore this connection. The function of this note is to present a short proof of one striking byproduct of this exploration.

The following results show that Theorem 1.1 has consequences in the Baire category and measurable settings:

Theorem 1.2 (Essentially [CM16]). *Let G be a locally finite Borel graph on a Polish space X, and μ a Borel probability measure on X*

(1) *There is a G-invariant comeager Borel set $C \subset X$ such that* $\operatorname{asi}(G \restriction C) \leq 1$.
(2) *If G is μ-hyperfinite, there is a G-invariant μ-conull Borel set $C \subset X$ such that* $\operatorname{asi}(G \restriction C) \leq 1$.

Thus, for example, we get a $\Delta + 2$ bound on Baire measurable edge chromatic numbers of simple graphs, and likewise for measure edge chromatic numbers of hyperfinite graphs.

For simple graphs, the best existing bound in these settings seems to have been $\Delta + O((\log^3 \Delta)\sqrt{\Delta})$. This was pointed out to us by Anton Bernshteyn; It follows from a recent randomized distributed algorithm for edge coloring [CHL+19] and a theorem of his linking such algorithms and measurable colorings [Ber20]. It should be noted though that this bound holds in the measurable setting even without an assumption of hyperfiniteness. For multigraphs Matt Bowen has shown (personal communication) that $\chi'_{BM}(G) \leq \lceil \frac{3\Delta}{2} \rceil + 1$, and likewise for measure chromatic numbers with hyperfiniteness. This is close to best possible in general, but is improved by our bound when Δ is much greater than the maximum edge multiplicity.

2. Proofs

For G a multigraph, U a set of vertices and $r \in \omega$, define $B_G(U, r)$ to be the set of vertices with path distance from U at most r. Also let $N_G(U)$ denote the set of vertices adjacent to some vertex in U. The following is essentially Lemma 2.1 from [BW21]:

Lemma 2.1. *Let G be a locally finite Borel multigraph on X with* $\operatorname{asi}(G) \leq s \in \omega$. *For any $n \in \omega$, we can find Borel sets $S_i^j \subset X$ for $j < s$ and $i < n$ such that, letting $S_i = \bigcup_j S_i^j$,*

(1) *The restrictions of G to $B_G(S_i^j, 3)$ for each i, j and to $(X \setminus S_i)$ for each i are component finite.*
(2) *For each j and $i \neq i'$, the path distance from S_i^j to $S_{i'}^j$ is at least 6.*

The main combinatorial tool in our construction is the following form of the Vizing adjacency lemma for multigraphs ([EFK84], Theorem 6):

Theorem 2.2. *Let G be a multigraph with maximum degree at most $k \in \omega$, e an edge between vertices x and y. Letting $\mu_G(x', y')$ denote the number of edges in G between vertices x' and y', suppose*

(1) $\chi'(G - \{e\}) \leq k$
(2) *For every* $z \in N_G(\{x\})$, $\deg_G(z) \leq k - \mu_G(x, z) + 1$.
(3) *The number of* $z \neq y$ *for which equality holds in the previous condition is at most* $k - \deg_G(y) - \mu_G(x, y) + 1$.

Then also $\chi'(G) \leq k$.

We will use the following consequence of this theorem, which is an analog of Lemma 4.0 from [Kie81] for multigraphs:

Lemma 2.3. *Let* G *be a multigraph on* X *with maximum degree at most* $k \in \omega$, *maximum edge multiplicity at most* $p \in \omega$, *and* $U \subset X$. *Suppose*

(1) $\chi'(G \restriction (X \setminus U)) \leq k$.
(2) *For every* $y \in N_G(U)$, $\deg_G(y) \leq k - p$.

Then also $\chi'(G) \leq k$.

Proof. We add one edge from $G \setminus (G \restriction (X \setminus U))$ at a time, using Theorem 2.2 each time to argue that the edge chromatic number stays below k. Indeed, if e is an added edge and we apply the theorem with x an endpoint of e in U, then for all $z \in N_G(\{x\})$, the inequality in (2) of the theorem is strict by hypothesis (2) of this lemma. Thus (2) and (3) from the theorem hold, as desired. □

We can now describe the inductive step in our construction:

Lemma 2.4. *Let* G *Borel multigraph graph on* X *with maximum edge multiplicity at most* $p \in \omega$, *and* $\chi'(G) \leq k + 1 \in \omega$. *Let* $S^j \subset X$ *Borel for* $j < s$ *such that, letting* $S = \bigcup_j S^j$, *the restrictions of* G *to* $X \setminus S$ *and to* $B_G(S^j, 3)$ *for each* j *are component finite. Then there are Borel matchings* M *and* N_i^j *for* $i < p$, $j < s$ *such that each edge of each* N_i^j *is contained in* $B_G(S^j, 3)$, *and letting* $G' = G \setminus (M \cup \bigcup_{i,j} N_i^j)$, $\chi'(G') \leq k$.

Proof. Consider the component finite bipartite Borel graphs $G \restriction B_G(S^j, 3)$ for $j < s$. By hypothesis, each component of each of these graphs admits a $k+1$-coloring. By the Lusin-Novikov uniformization theorem, there is a Borel way of picking such a coloring for each component. Thus, let $d^j : G \restriction B_G(S^j, 3) \to k+1$ be a Borel $k+1$-coloring for each j. For each j and $i < p$, let $N_i^j = (d^j)^{-1}(\{i\})$. Let $G^* = G \setminus \bigcup_{i,j} N_i$.

$G^* \restriction (X \setminus S)$ is also component finite since $G^* \subset G$, so by the same argument, let $G^* \restriction (X \setminus S) \to k+1$ be a Borel $k+1$-coloring. Then $c^{-1}(\{0\})$ is a Borel independent set of edges, so by [KST99], it is contained in some maximal Borel independent set of edges, call it M. Then $G' = G^* \setminus M$.

It remains to check $\chi'(G') \leq k$. First note $\Delta(G') \leq k$: If $x \in B_G(S^j, 2)$ for some j, then $B_G(\{x\}, 1) \subset B_G(S^j, 3)$, so d^j gave a $k+1$-coloring of the edges with x as an endpoint. We removed p color sets in the construction of G^*, though, so in fact

$$\deg_{G'}(x) \leq \deg_{G^*}(x) \leq k - p + 1 \leq k.$$

Else, $B_G(\{x\}, 1) \subset X \setminus S$, so the same argument with c in place of d^j works.

Let $W = \{x \in B_G(S, 2) \mid \deg_{G'}(x) > k - p\}$. We wish to apply Lemma 2.3 with $U = V := W \cap B_G(S, 1)$. For condition 2 from the lemma, it suffices to show W is G'-independent. Suppose to the contrary $x, y \in W$ are adjacent. By the inequality in the previous paragraph, we must then have $\deg_{G*}(x) = \deg_{G^*}(y) = k - p + 1$. In the construction of G' from G^*, though, we removed a maximal matching, and so the degree of at least one of x and y must have dropped by one, contradicting the definition of W.

For condition 1, we need to see $\chi'(G' \upharpoonright (X \setminus V)) \leq k$. Call this graph H. We will apply Lemma 2.3 to H with $U = S \setminus W$. For condition 2 from the lemma, note that if $y \in N_H(S \setminus W)$, $y \in B_G(S, 1) \setminus W$, so $\deg_H(y) \leq \deg_{G'}(y) \leq k - p$ by definition of W. Condition 1 of the lemma holds since c was a $k+1$-coloring of $G^* \upharpoonright (X \setminus S)$ and we removed a color in passing to G'. □

We now prove Theorem 1.1:

Proof. Apply Lemma 2.1 with $n = \chi'(G)$ to get Borel sets $S_i^j \subset X$ for $j < s$ and $i < \chi'(G)$. Let $G_0 = G$.

Suppose $i < \chi'(G)$ and we have Borel $G_i \subset G$ with $\chi'(G_i) \leq \chi'(G) - i$. Apply Lemma 2.4 to G_i and the S_i^j's to get Borel matchings M_i and $N_{i,l}^j$ for $l < p$, $j < s$ so that each edge of each $N_{i,l}^j$ is contained in $B_G(S_i^j, 3)$ and, letting $G_{i+1} = G_i \setminus (M_i \cup \bigcup_{i,j} N_{i,l}^j)$, $\chi'(G_{i+1}) \leq \chi'(G) - (i+1)$. This last condition lets us continue the construction.

At the end, $G_{\chi'(G)}$ has edge chromatic number at most $\chi'(G) - \chi'(G) = 0$, and so is empty. Therefore the matchings we removed along the way union to all of G. Furthermore, for each fixed $j < s$ and $l < p$, $\bigcup_i N_{i,l}^j$ is still a matching. This because each edge of each $N_{i,l}^j$ is contained in $B_G(S_i^j, 3)$, and by condition 2 from Lemma 2.1, these sets are disjoint for different i. Therefore we have $\chi'(G) + ps$ Borel matchings which union to G: They are M_i for $i < \chi'(G)$ and $\bigcup_i N_{i,l}^j$ for $j < s$, $l < p$. □

References

[Ber20] Anton Bernshteyn, *Distributed algorithms, the lovász local lemma, and descriptive combinatorics*, preprint, arXiv:2004.04905.

[BW21] Matt Bowen and Felix Weilacher, *Definable könig theorems*, 2021, preprint, arXiv:2112.10222.

[CHL⁺19] Yi-Jun Chang, Qizheng He, Wenzheng Li, Seth Pettie, and Jara Uitto, *Distributed edge coloring and a special case of the constructive lovász local lemma*, ACM Trans. Algorithms **16** (2019), no. 1.

[CJM⁺20] Clinton Conley, Steve Jackson, Andrew Marks, Brandon Seward, and Robin Tucker-Drob, *Borel asymptotic dimension and hyperfinite equivalence relations*, 2020, preprint, arXiv:2009.06721.

[CM16] Clinton T. Conley and Benjamin D. Miller, *A bound on measurable chromatic numbers of locally finite Borel graphs*, Math. Res. Lett. **23** (2016), no. 6, 1633–1644.

[EFK84] A. Ehrenfeucht, V. Faber, and H.A. Kierstead, *A new method of proving theorems on chromatic index*, Discrete Mathematics **52** (1984), no. 2, 159–164.

[Kie81] Henry A. Kierstead, *Recursive colorings of highly recursive graphs*, Canadian Journal of Mathematics **33** (1981), no. 6, 1279–1290.

[KST99] A. S. Kechris, S. Solecki, and S. Todorcevic, *Borel chromatic numbers*, Adv. Math. **141** (1999), no. 1, 1–44.