Differential forms (Spl Lecture)

· Primitive forms on $\mathbb{R}^3$. Coordinates (x_1, x_2, x_3) and (x, y, z) interchangeably.

Def^n 1) A primitive 1-form on $\mathbb{R}^3$ is defined as either dx_1, dx_2, dx_3.

2) A primitive p-form is an expression of form

$$\omega = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}.$$

where $i_1, \ldots, i_p \in \{1, 2, 3\}$ with the rules

(R1) $dx_{i_1} \wedge \cdots \wedge dx_{i_p} = 0 \iff i_j = i_k$ for some $j \neq k$

(R2) $\cdots \wedge dx_i \wedge dx_j \wedge \cdots = -(\cdots \wedge dx_j \wedge dx_i \wedge \cdots)$

i.e switching two consecutive primitive 1-forms, one incurs a $-$ve sign.

(R3) (Wedge product) $\wedge : \{p\text{-form}\} \times \{q\text{-form}\} \longrightarrow \{(p+q)\text{-form}\}$

$$(dx_{i_1} \wedge \cdots \wedge dx_{i_p}) \wedge (dx_{j_1} \wedge \cdots \wedge dx_{j_q})$$
$$= dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}$$

Consequences 1). Any primitive p-form can be written uniquely as

$$\omega = \pm\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}$$

$i_1 < i_2 < i_3 \cdots < i_p$.

2) No non-zero p-form if $p \geq 4$.

3) If ω primitive p-form & η prim. q-form.
Then
$$\omega \wedge \eta = (-1)^{pq}\, \eta \wedge \omega.$$

e.g: $\omega = dx_1$, $\eta = dx_2$, $\omega \wedge \eta = dx_1 \wedge dx_2$

$$= -dx_2 \wedge dx_1 = \eta \wedge \omega.$$

Here $(-1)^{pq} = -1$.

$\omega = dx_1$, $\eta = dx_2 \wedge dx_3$, $\omega \wedge \eta = dx_1 \wedge dx_2 \wedge dx_3$

$$\eta \wedge \omega = dx_2 \wedge dx_3 \wedge dx_1 = -dx_2 \wedge dx_1 \wedge dx_3$$
$$= dx_1 \wedge dx_2 \wedge dx_3 = \omega \wedge \eta.$$

Here $pq = 2 \Rightarrow (-1)^{pq} = 1$.

• Differential forms on $\Omega \subset \mathbb{R}^3$. Let Ω be open, connected

Def^n: 1) A 0-form on Ω is a function $f : \Omega \to \mathbb{R}$

2) A p-form on Ω is an expression
$$\omega = \sum_{i_1 < \cdots < i_p} f_{i_1 i_2 \cdots i_p} \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

each $f_{i_1 \cdots i_p} : \Omega \longrightarrow \mathbb{R}$. called $(i_1, \ldots, i_p)^{th}$ component

We say ω is a <u>smooth</u> p-form if all component functions have partial derivatives of all orders, and set

$$\Lambda^k(\Omega) = \{ \text{set of smooth p-forms on } \Omega \}.$$

<u>Examples</u> 1) In $\mathbb{R}^3$ with (x, y, z).

<u>0-form</u>: $f : \Omega \to \mathbb{R}$.

<u>1-form</u>: $f\,dx + g\,dy + h\,dz$, $f, g, h : \Omega \to \mathbb{R}$.

<u>2-form</u>: $f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy$.

<u>3-form</u>: $f\,dx \wedge dy \wedge dz$.

Written this way, there is a nice duality. Namely we have a map $* : \Lambda^k(\Omega) \to \Lambda^{3-k}(\Omega)$. s.t $** = id$.

$$*(f) = f\,dx \wedge dy \wedge dz$$

$$*(f\,dx + g\,dy + h\,dz) = f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy.$$

2) Given v.f $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$, there is a corresponding 1-form.

$$(\vec{F})^* = \omega_{\vec{F}} = P\,dx + Q\,dy + R\,dz.$$

<u>Def^n</u> (Wedge prod. of forms) ① If $f, g: \Omega \to \mathbb{R}$ & ω, η are <u>primitive</u> p and q - forms, then define

$$f\omega \wedge g\eta = fg \cdot \omega \wedge \eta .$$

② If $f_1, f_2, g_1, g_2 \; \Omega \to \mathbb{R}$ and ω_1, ω_2 p-forms; $\eta_1 \& \eta_2$ are q - forms, Then

$$(f_1\omega_1 + f_2\omega_2) \wedge (g_1\eta_1 + g_2\eta_2)$$
$$= f_1 g_1 \, \omega_1 \wedge \eta_1 + f_1 g_2 \, \omega_1 \wedge \eta_2$$
$$+ f_2 g_1 \, \omega_2 \wedge \eta_1 + f_2 g_2 \, \omega_2 \wedge \eta_2 .$$

<u>Ex</u>: 1) $\omega = x^2 dx + x\, dy$, $\eta = x\, dx \wedge dz + y\, dy \wedge dz$.

$$\omega \wedge \eta = x^2 dx \wedge dy \wedge dz + x\, dy \wedge dx \wedge dz$$
$$= (x^2 - x)\, dx \wedge dy \wedge dz .$$

2) If $\vec{F} \;\&\; \vec{G}$ are v.f. on Ω.

<u>Check</u>: $\boxed{\omega_{\vec{F}} \wedge \omega_{\vec{G}} = \omega_{\vec{F} \times \vec{G}}}$

So wedge-prod. is a generalization of <u>cross</u> product

Def^n: Given an $\omega \in \Lambda^3(\Omega)$, we define its *integral* on Ω by

$$\int_{\Omega} \omega := \underbrace{\iiint_{\Omega} f \, dV}_{}$$

Usual triple integral.

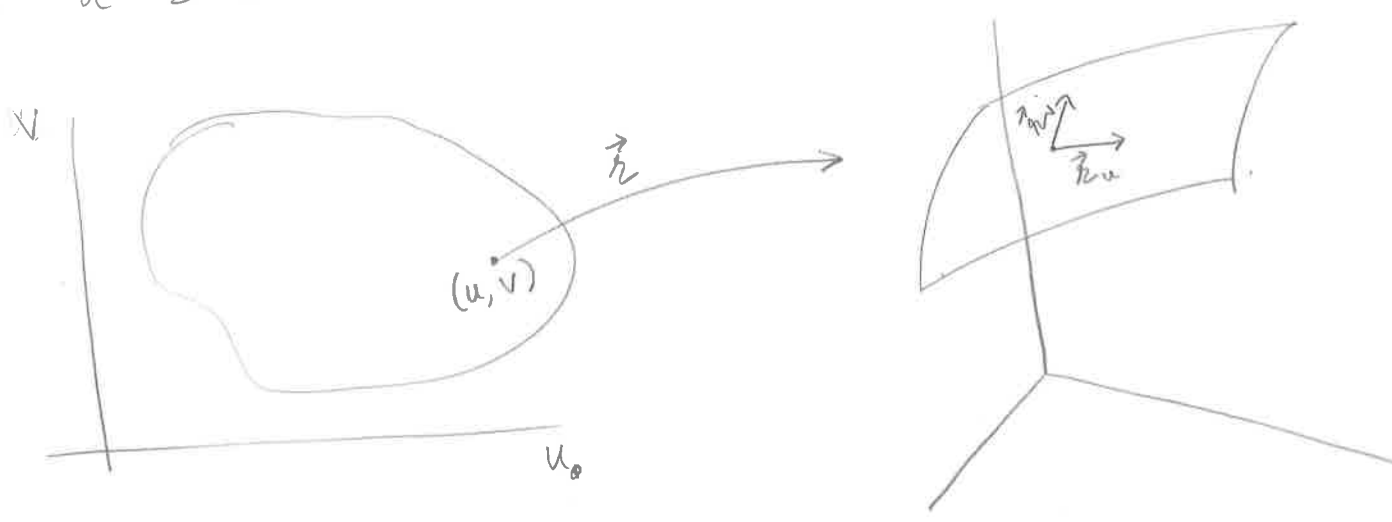
where $\omega = f \, dx \wedge dy \wedge dz$.

Similarly, one could define p-forms on $\mathbb{R}^n$ for $n = 1, 2$, and define integrals of p-forms on $\Omega \subset \mathbb{R}^p$. i.e.

$$\boxed{p\text{- form can be integrated on a region in } \mathbb{R}^p.}$$

- Integration on k-dim manifolds.

Recall a surface is



Def^n: A pair $(E, \vec{r})$ is said to parametrize a p-dimensional manifold if.

① $E \subset \mathbb{R}^p$ open, connected.

② $\vec{r} : E \longrightarrow \mathbb{R}^3$ with smooth components.

③ (non-deg.) $(\vec{r}_{u_1})^\# \wedge (\vec{r}_{u_2})^\# \wedge \cdots \wedge (\vec{r}_{u_p})^\# \neq 0$

at any point. Here

$$\vec{r}_{u_j} = \frac{\partial \vec{r}}{\partial u_j}.$$

The image $\vec{r}(E) := M$ is called a p-dim. manifold. We write $x_j = x_j(u_1, \ldots, u_p)$ on M.

<u>Def</u>

__Def^n__ (Pull back). Let Ω be a nbd of M in $\mathbb{R}^3$. One can define $\vec{r}^* : \Lambda^q(\Omega) \longrightarrow \Lambda^q(E)$. $(q \leq p)$.

① $$\vec{r}^*(dx_j) = \frac{\partial x_j}{\partial u_k} \cdot du_k$$

$$\vec{r}^*(f\,dx_j) = f(\vec{r}(u_1, \ldots, u_p)) \cdot \frac{\partial x_j}{\partial u_k} \cdot du_k$$

② $$\vec{r}^*(\omega \wedge \eta) = \vec{r}^*(\omega) \wedge \vec{r}^*(\eta).$$

__Prop.__ If $i_1 < i_2 < \cdots < i_q$, then

Jacobian

$$\vec{r}^*(dx_{i_1} \wedge \cdots \wedge dx_{i_q}) = \frac{\partial(x_{i_1}, \ldots, x_{i_q})}{\partial(u_{j_1}, \ldots, u_{j_q})} \cdot du_{j_1} \wedge \cdots \wedge du_{j_q}$$

Ex: 1) $\underline{p=1}$: M is a curve. $\vec{r}: [a,b] \longrightarrow \mathbb{R}^3$.

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Then
$$\vec{r}^*(dx) = x'(t)\,dt.$$
$$\vec{r}^*(dy) = y'(t)\,dt.$$
$$\vec{r}^*(dz) = z'(t)\,dt.$$

This is what we actually meant when we loosely wrote $dx = x'(t)\,dt$ before.

2) $\underline{p=2}$: M is a surface.

Def^n: Given a <u>p-form</u> in nbd of <u>p-dim</u> manifold. M $\subset \mathbb{R}^3$, define the <u>integral on M</u> by

$$\int_M \omega := \int_E \vec{r}^*\omega.$$

Ex: 1) $\vec{F}$ v.f in nbd of a <u>curve</u> $\vec{r}(t)$.

Consider $\omega_{\vec{F}} = P\,dx + Q\,dy + R\,dy$.

$$\vec{r}^*(\omega_{\vec{F}}) = P \cdot x'(t)\,dt + Q\,y'(t)\,dt + R\,z'(t)\,dt.$$

So
$$\int_M \omega := \int_a^b P x'\,dt + Q y'\,dt + R z'\,dt$$
$$= \int_M \vec{F} \cdot d\vec{r}.$$

Recovers the usual line integral.

2) $M \subset \mathbb{R}^3$ surface, $\vec{F}$ a v.f in a nbd of M. Let $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ Then

$$*(\omega_{\vec{F}}) = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy.$$

So,

$$\vec{r}^*(*\omega_{\vec{F}}) = \left[P\frac{\partial(y,z)}{\partial(u,v)} + Q\frac{\partial(z,x)}{\partial(u,v)} + R\frac{\partial(x,y)}{\partial(u,v)}\right] du \wedge dv.$$

On the other hand:

$$\vec{r}_u \times \vec{r}_v = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix}$$

$$= \frac{\partial(y,z)}{\partial(u,v)}\hat{i} + \frac{\partial(z,x)}{\partial(u,v)}\hat{j} + \frac{\partial(x,y)}{\partial(u,v)}\hat{k}$$

So $$\vec{r}^*(*\omega_{\vec{F}}) = [\vec{F} \cdot (\vec{r}_u \times \vec{r}_v)]\, du \wedge dv.$$

$$\Rightarrow \int_M *\omega_{\vec{F}} := \int_E \vec{r}^*(*\omega_{\vec{F}})$$

$$= \int_E \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, du \wedge dv$$

$$= \iint_M \vec{F} \cdot d\vec{S}$$

- Exterior derivative.

Def^n: Let $\Omega \subset \mathbb{R}^3$. The exterior derivative is a map $d: \Lambda^p(\Omega) \longrightarrow \Lambda^{p+1}(\Omega)$.

0-form: $df := \dfrac{\partial f}{\partial x}\,dx + \dfrac{\partial f}{\partial y}\,dy + \dfrac{\partial f}{\partial z}\,dz$.

1-form: $d(P\,dx + Q\,dy + R\,dz)$

$$= dP \wedge dx + dQ \wedge dy + dR \wedge dz$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx$$

$$+ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

2-form: $d(P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy)$.

$$= dP \wedge dy \wedge dz + dQ \wedge dz \wedge dx + dR \wedge dx \wedge dy$$

$$= \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz$$

Rk ① $d(f) = (\vec{\nabla} f)^{\#}$

$d(\omega_{\vec{F}}) = *(\vec{\nabla} \times \vec{F})^{\#}$

$d(*\,\omega_{\vec{F}}) = *\,(\mathrm{div}\,\vec{F})$.

② One can show, for any $\omega \in \Lambda^p(\Omega)$,

$\boxed{d(d\omega) = 0}$ (we write $d^2 = 0$).

This <u>encapsulates</u> the identities

$$\text{curl}\, \vec{\nabla} f = \vec{0}, \quad \text{div}(\text{curl}\, \vec{F}) = 0.$$

• <u>Stokes' theorem</u>.

<u>Thm</u>: Let M be a p-dim manifold s.t ∂M be a $(p-1)$ dimensional manifold. If ω is a $(p-1)$ form. Then

$$\boxed{\int_{\partial M} \omega = \int_M d\omega.}$$

<u>Ex</u>: ① If $p = 2$. Then M is a surface & ∂M is a boundary curve. Sps $\vec{F}$ is a v.f on a nbd of M. We have seen

$$\int_{\partial M} \omega_{\vec{F}} = \int_{\partial M} \vec{F} \cdot d\vec{r}.$$

On the other hand, $d\omega_{\vec{F}} = *(\vec{\nabla} \times \vec{F})^{\#}$, and we have seen

$$\int_M *(\vec{\nabla} \times \vec{F})^{\#} = \int_M (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

So we recover

$$\int_{\partial M} \vec{F} \cdot d\vec{r} = \iint_{M} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}.$$

(2) $p = 3$. M is a region in $\mathbb{R}^3$ & ∂M is a closed surface. Let $\omega = * \omega_{\vec{F}}$. Then.

$$\int_{\partial M} * \omega_{\vec{F}} = \int_{\partial M} \vec{F} \cdot d\vec{S}$$

$$\int_{M} d * \omega_{\vec{F}} = \int_{M} * (\operatorname{div} \vec{F}).$$

$$= \int_{M} (\operatorname{div} \vec{F}) \, dx \wedge dy \wedge dz$$

$$:= \iiint_{\Omega} (\operatorname{div} \vec{F}) \, dV.$$