MATH H53 : PRACTICE PROBLEMS FOR MID-TERM-2

1. Analytic Geometry

- Chapter-12 Review (pg. 834) - 4, 7, 11, 23, 27

2. Curves in 2 and 3 dimensions

- Chapter-10 Review - 11, 15, 29, 31, 39, 57
- Chapter-13 Review - 9, 11, 20, 21

Additional Problems.

(1) Show that any tangent line to the hyperbola $x^2 - y^2 = c$ touches the hyperbola halfway between the points of intersection of the tangent and the asymptotes.

(2) Show that the curve with vector equation

$$\mathbf{r}(t) = \langle a_1 t^2 + b_1 t + c_1, a_2 t^2 + b_2 t + c_2, a_3 t^2 + b_3 t + c_3 \rangle$$

lies in a plane, and find the equation of that plane.

(3) If a point moves so that the velocity and acceleration have constant magnitudes $||\mathbf{v}||$ and $||\mathbf{a}||$ respectively, then show that the curvature of the trajectory of the path is also a constant, namely

$$\kappa = ||\mathbf{a}|| \Big/ ||\mathbf{v}||^2.$$

3. Limits, Continuity and Differentiation

- Chapter 14 Review - 27, 47

Additional Problems.

(1) Prove the following -

(a) If $z = f(x - y)$ then $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

(b) If f is homogenous of degree n (that is, $f(tx, ty) = t^n f(x, y)$), then

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n-1) f(x, y).$$

(c) If $\mathbf{F}$ is a *radial* vector field on $\mathbb{R}^3 \setminus (0, 0, 0)$ given by $\mathbf{F} = f(r)\mathbf{r}$ where as usual $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = \mathbf{r}$, then

$$\mathrm{div}\mathbf{F} = 3\mathrm{f(r)} + \mathrm{f'(r)r}.$$

Hence show that the only divergence free radial vector field as above is $\mathbf{F} = k\mathbf{r}/r^3$ for some constant k.

(2) Find the points on the curve of intersection of the two surfaces

$$x^2 - xy + y^2 - z^2 = 1 \text{ and } x^2 + y^2 = 1$$

which are nearest to the origin.

(3) Determine all the local and absolute extremums and saddle points for the function $f(x, y) = xy(1 - x^2 - y^2)$ on the square $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

4. Integration

- Chapter 15 Review - 21, 26, 31, 33

Additional Problems.

(1) Evaluate
$$\iint_D \frac{y^2}{x}\,dx\,dy,$$
where D is the region between the parabolas $x = (1 - y^2)$ and $x = 3(1 - y^2)$ using the change of variables $x = v(1 - u^2)$ and $y = u$.

(2) Find the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 5$ and below by the paraboloid $x^2 + y^2 = 4z$.

(3) Find the volume bounded by the xy-plane, the cylinder $x^2 + y^2 = 2x$ and the cone $z = \sqrt{x^2 + y^2}$.

(4) Evaluate
$$\iiint_E \frac{1}{(1 + x^2 + y^2 + z^2)^3}\,dV,$$
where E is the solid above the cone $3z^2 = x^2 + y^2$ and below the sphere $x^2 + y^2 + z^2 = 1$.

5. Vector Calculus

- Chapter 16 Review - 7, 11, 13, 19, 25, 37

Additional Problems.

(1) Show that the area of the portion of the conical surface $x^2 + y^2 = z^2$ that lies between planes $z = 0$ and $x + 2z = 3$ is $2\pi\sqrt{6}$.

(2) Show that the area of the portion of the conical surface $x^2 + y^2 = z^2$ that is above the xy-plane and cut off by the sphere $x^2 + y^2 + z^2 = 2ax$ is $\sqrt{2}\pi a^2/4$.

(3) Compute the flux of $\mathbf{F} = x\mathbf{i} - (2x + y)\mathbf{j} + z\mathbf{k}$ across the hemisphere $x^2 + y^2 + z^2 = 1, z > 0$.

6. Green's, Stokes' and Gauss' Theorems

- Chapter 16 Review - 29, 33, 38, 39, 41

(1) If f and g have continuous first partials in an open set $D \subset \mathbb{R}^2$ with boundary a simple closed curve C. Then show that
$$\int_C f\nabla g \cdot\, d\mathbf{r} = -\int_C g\nabla f \cdot\, d\mathbf{r}$$

(2) Show that
$$\int_C y\,dx + z\,dy + x\,dz = \pm\pi a^2\sqrt{3},$$
where C is the intersection of $x^2 + y^2 + z^2 = a^2$ and the plane $x + y + z = 0$. What is the orientation that gives a positive circulation?

(3) Show that
$$\int_C (y - z)\,dx + (z - x)\,dy + (x - y)\,dz = -2\pi a(a + b),$$
where C is the intersection of the cylinder $x^2 + y^2 = a^2$ and the plane $x/a + z/b = 1$, and the orientation is anti-clockwise when viewed from above.

(4) Recall that the *Laplacian* of φ is defined as

$$\Delta\varphi = \operatorname{div}(\nabla\varphi).$$

Let S denote a unit sphere centered at the origin, $\hat{\mathbf{n}}$ be its outward unit normal and let B denote the enclosed ball of radius one.

(a) Show that

$$\iiint_B \Delta\varphi \, dV = \iint_S \nabla\varphi \cdot \hat{\mathbf{n}} \, dS.$$

(b) Suppose a scalar field φ which is never zero has the property that

$$||\nabla\varphi||^2 = \varphi \text{ and } \operatorname{div}(\varphi\nabla\varphi) = 2\varphi.$$

Evaluate the surface integral

$$\iint_S \nabla\varphi \cdot \hat{\mathbf{n}}. \, dS.$$